

Construction of Arbitrary Order Conformally Invariant Operators in Higher Spin Spaces

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Abstract

This paper completes the construction of arbitrary order conformally invariant differential operators in higher spin spaces. Jan Slovák has classified all conformally invariant differential operators on locally conformally flat manifolds. We complete his results in higher spin theory in Euclidean space by giving explicit expressions for arbitrary order conformally invariant differential operators, where by conformally invariant we mean equivariant with respect to the conformal group of S^m acting in Euclidean space \mathbb{R}^m . We name these the fermionic operators when the order is odd and the bosonic operators when the order is even. Our approach explicitly uses convolution type operators to construct conformally invariant differential operators. These convolution type operators are examples of Knapp-Stein operators and they can be considered as the inverses of the corresponding differential operators. Intertwining operators of these convolution type operators are provided and intertwining operators of differential operators follow immediately. This reveals that our convolution type operators and differential operators are all conformally invariant. This also gives us a class of conformally invariant convolution type operators in higher spin spaces. Their inverses, when they exist, are conformally invariant pseudo-differential operators. Further we use Stein Weiss gradient operators and representation theory for the Spin group to naturally motivate the construction of Rarita-Schwinger operators.

Keywords: Fermionic operators, Bosonic operators, Conformal invariance, Fundamental solutions, Intertwining operators, Convolution type operators, Knapp-Stein operators.

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1 Introduction

The *higher spin theory* in Clifford analysis was first introduced with the Rarita-Schwinger operators [6]. This theory considers generalizations of classical Clifford analysis techniques to higher spin spaces [3, 5, 6, 13, 16, 23], focusing on operators acting on functions on \mathbb{R}^m that take values in arbitrary irreducible representations of $Spin(m)$. Generally these are polynomial representations, such as spaces of k -homogeneous monogenic or harmonic polynomials (\mathcal{M}_k or \mathcal{H}_k) corresponding to particles of half-integer spin or integer spin. Here monogenic functions are solutions to the Euclidean Dirac equation.

After the Laplacian was pointed out no longer to be conformally invariant in higher spin space, Eelbode and Roels [16] constructed a second order conformally invariant operator: the (generalized) Maxwell operator acting on $C^\infty(\mathbb{R}^m, \mathcal{H}_1)$, where the target space \mathcal{H}_1 is a degree-1 homogeneous harmonic polynomial space. De Bie and his co-authors [3] then generalized this Maxwell operator to the case when it acts on $C^\infty(\mathbb{R}^m, \mathcal{H}_k)$. This is what they call the higher spin Laplace operator. In [11], we introduced fermionic operators and bosonic operators as the generalization of k -th powers of the Euclidean Dirac operator to higher spin space, considering the special case of target space of degree-1 homogeneous polynomials while using similar techniques as in [3, 16]. The connections to mathematical physics emphasized in that work also apply to the present manuscript. We later constructed the 3rd order fermionic and 4th order bosonic operators when the target space is a degree- k homogeneous polynomial space [10]. Unfortunately, the generalized symmetry approach we used in [10, 11] was computationally infeasible for arbitrary higher order conformally invariant operators.

The methods we use to construct conformally invariant operators are usually either of the following type.

1. Verify some differential operator is conformally invariant under Möbius transformations with the help of the Iwasawa decomposition of the Möbius transformation, for instance as in [13].
2. Show the generalized symmetries of some differential operator generate a conformal Lie algebra, for instance as in [3, 16].

This paper uses a method different from these. We start by applying Slovák [34] and Souček's [35] results with arguments of Bureš et al. [6] to get fundamental solutions of arbitrary order conformally invariant differential operators in higher spin spaces. Then we only need to construct differential operators with those specific fundamental solutions. In particular, from the fundamental solutions of first and second order conformally invariant differential operators obtained from the preceding argument, we can also find the Rarita-Schwinger operators [6] and higher spin Laplace operators [3] by verifying they have such fundamental solutions. Arguing by induction, we then complete the work on

constructing conformally invariant operators in higher spin spaces by providing explicit forms of arbitrary j -th order conformally invariant operators in higher spin spaces with $j > 2$.

Notably, we discover a new analytic approach to show that a differential operator is conformally invariant. More specifically, we use its fundamental solution to define a convolution type operator and then the fundamental solution can be realized as the inverse of the corresponding differential operator in the sense of such convolution. Hence, if we can show the fundamental solution (as a convolution operator) is conformally invariant, then as the inverse, the corresponding differential operator will also be conformally invariant. Thus the intertwining operators of the fundamental solution (as a convolution operator) are the inverses of the intertwining operators of the differential operators. This idea brings us an infinite class of conformally invariant convolution type operators in higher spin spaces; their inverses, when they exist, are generalized conformally invariant pseudo-differential operators. More details can be found in Section 4.1. It is worth pointing out that these intertwining operators and convolution type operators are special cases of Knapp-Stein intertwining operators and Knapp-Stein operators in higher spin theory ([7, 20]).

Our study of conformally invariant differential operators in higher spin spaces suggests a distinct Representation-Theoretic approach to Clifford analysis, in contrast to the classical Stokes approach. In the latter approach, the motivation for Dirac-type operators is to obtain operators satisfying a Stokes-type theorem. This does not need irreducible representation theory. In contrast, in the Representation-Theoretic approach, we consider functions taking values in irreducible representations of the Spin group. This forces one to consider irreducible representation theory, as happens elsewhere in the literature where Dirac operators are used [19] and especially in spin geometry [22]. Moreover, irreducible spin representations are natural for studying spin invariance and in particular conformal invariance. That is not to dismiss the Stokes approach—it is used, for instance, to establish the L^2 boundedness of the double layer potential operator on Lipschitz graphs [24], and other applications are found in such works as [4]. Though the present work aims to demonstrate the value of the Representation-Theoretic approach, in future work the two distinct approaches may complement each other.

The paper is organized as follows. We briefly introduce Clifford algebras, Clifford analysis, and representation theory of the Spin group in Section 2. We recall the Stein-Weiss construction of the Euclidean Dirac operator and Rarita-Schwinger operator from [9, 33] in Section 3. This motivates the extensive use of representation theory in our recent work on conformally invariant differential operators in higher spin theory. Further, this construction also reveals that Stein-Weiss gradient operators and representation theory of the Spin group provide the most natural approach to the study of Rarita-Schwinger operators.

In Section 4, we provide an approach different from [3, 16] to construct these confor-

mally invariant differential operators. This approach relies heavily on the fundamental solutions of these conformally invariant differential operators. We also define a convolution type operator associated to each fundamental solution to show each fundamental solution is actually the inverse of the corresponding differential operator. An explicit proof for the intertwining operators of these convolution type operators is provided there. This implies conformal invariance of these convolution type operators and conformal invariance of the corresponding differential operators is shown immediately. We point out that this idea also gives an infinite class of conformally invariant convolution type operators; their inverses, when they exist, are generalized conformally invariant pseudo-differential operators. We also show that the higher spin Laplace operators [3] can also be derived from this approach. Then we introduce bosonic operators \mathcal{D}_{2j} as the generalization of D_x^{2j} when acting on $C^\infty(\mathbb{R}^m, \mathcal{H}_k)$ and fermionic operators \mathcal{D}_{2j-1} as the generalization of D_x^{2j-1} when acting on $C^\infty(\mathbb{R}^m, \mathcal{M}_k)$, where D_x is the Euclidean Dirac operator with respect to the variable x . The connections between these and lower order conformally invariant operators are also revealed in the construction. Moreover, since the construction is explicitly based on the uniqueness of the operators and their fundamental solutions with the appropriate intertwining operators for a conformal transformation, the conformal invariance and fundamental solutions of the bosonic and fermionic operators arise naturally in our formalism.

We cover technical details and proofs for the fermionic case in Section 5.

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2 Preliminaries

2.1 Clifford algebra

A real Clifford algebra, \mathcal{Cl}_m , can be generated from the m -dimensional real Euclidean space \mathbb{R}^m by considering the relationship

$$\underline{x}^2 = -\|\underline{x}\|^2$$

for each $\underline{x} \in \mathbb{R}^m$. We have $\mathbb{R}^m \subseteq \mathcal{Cl}_m$. If $\{e_1, \dots, e_m\}$ is an orthonormal basis for \mathbb{R}^m , then $\underline{x}^2 = -\|\underline{x}\|^2$ tells us that

$$e_i e_j + e_j e_i = -2\delta_{ij},$$

where δ_{ij} is the Kronecker delta function. An arbitrary element of the basis of the Clifford algebra can be written as $e_A = e_{j_1} \cdots e_{j_r}$, where $A = \{j_1, \dots, j_r\} \subset \{1, 2, \dots, m\}$ and

$1 \leq j_1 < j_2 < \dots < j_r \leq m$. Hence for any element $a \in \mathcal{C}l_m$, we have $a = \sum_A a_A e_A$, where $a_A \in \mathbb{R}$. Similarly, the complex Clifford algebra $\mathcal{C}l_m(\mathbb{C})$ is defined as the complexification of the real Clifford algebra

$$\mathcal{C}l_m(\mathbb{C}) = \mathcal{C}l_m \otimes \mathbb{C}.$$

We consider the real Clifford algebra $\mathcal{C}l_m$ throughout this subsection, but in the rest of the paper we consider the complex Clifford algebra $\mathcal{C}l_m(\mathbb{C})$ unless otherwise specified. The complex Clifford algebra may be viewed as a vector space over the field of scalars \mathbb{C} ; correspondingly, in this work we may refer to complex-valued functions as scalar-valued functions. Alternatively, there is an isomorphic copy of \mathbb{C} embedded in $\mathcal{C}l_m(\mathbb{C})$ that may be considered as a scalar subspace.

For $a = \sum_A a_A e_A \in \mathcal{C}l_m$, we define the reversion of a as

$$\tilde{a} = \sum_A (-1)^{|A|(|A|-1)/2} a_A e_A,$$

where $|A|$ is the cardinality of A . In particular, $\widetilde{e_{j_1} \cdots e_{j_r}} = e_{j_r} \cdots e_{j_1}$. Also $\tilde{a}\tilde{b} = \tilde{\tilde{a}\tilde{b}}$ for $a, b \in \mathcal{C}l_m$.

The Pin and Spin groups play an important role in Clifford analysis. The Pin group can be defined as

$$Pin(m) = \{a \in \mathcal{C}l_m : a = y_1 y_2 \cdots y_p, y_1, \dots, y_p \in \mathbb{S}^{m-1}, p \in \mathbb{N}\},$$

where \mathbb{S}^{m-1} is the unit sphere in \mathbb{R}^m . $Pin(m)$ is clearly a multiplicative group in $\mathcal{C}l_m$, see [4] for more details.

Now suppose $a \in \mathbb{S}^{m-1} \subseteq \mathbb{R}^m$. If we consider axa , we may decompose

$$x = x_{a\parallel} + x_{a\perp},$$

where $x_{a\parallel}$ is the projection of x onto a and $x_{a\perp}$ is the remainder part of x perpendicular to a . Hence $x_{a\parallel}$ is a scalar multiple of a and we have

$$axa = ax_{a\parallel}a + ax_{a\perp}a = -x_{a\parallel} + x_{a\perp}.$$

So the action axa describes a reflection of x in the direction of a . By the Cartan-Dieudonné Theorem each $O \in O(m)$ is the composition of a finite number of reflections. If $a = y_1 \cdots y_p \in Pin(m)$, we define $\tilde{a} := y_p \cdots y_1$ and observe $ax\tilde{a} = O_a(x)$ for some $O_a \in O(m)$. Choosing y_1, \dots, y_p arbitrarily in \mathbb{S}^{m-1} , we have the group homomorphism

$$\theta : Pin(m) \longrightarrow O(m) : a \mapsto O_a,$$

with $a = y_1 \cdots y_p$ and $O_a x = ax\tilde{a}$ is surjective. Further $-ax(-\tilde{a}) = ax\tilde{a}$, so $1, -1 \in Ker(\theta)$. In fact $Ker(\theta) = \{1, -1\}$. See [26]. The Spin group is defined as

$$Spin(m) = \{a \in \mathcal{C}l_m : a = y_1 y_2 \cdots y_{2p}, y_1, \dots, y_{2p} \in \mathbb{S}^{m-1}, p \in \mathbb{N}\}$$

and it is a subgroup of $Pin(m)$. There is a group homomorphism

$$\theta : Spin(m) \longrightarrow SO(m)$$

that is surjective with kernel $\{1, -1\}$ and defined by the above group homomorphism for $Pin(m)$. Thus $Spin(m)$ is the double cover of $SO(m)$. See [26] for more details.

For a domain U in \mathbb{R}^m , a diffeomorphism $\phi : U \longrightarrow \mathbb{R}^m$ is said to be conformal if, for each $x \in U$ and each $\mathbf{u}, \mathbf{v} \in TU_x$, the angle between \mathbf{u} and \mathbf{v} is preserved under the corresponding differential at x , $d\phi_x$. For $m \geq 3$, a theorem of Liouville tells us the only conformal transformations are Möbius transformations. Ahlfors and Vahlen show any Möbius transformation on $\mathbb{R}^m \cup \{\infty\}$ can be expressed as $y = (ax + b)(cx + d)^{-1}$ with $a, b, c, d \in \mathcal{Cl}_m$ satisfying the following conditions [1]:

1. a, b, c, d are all products of vectors in \mathbb{R}^m .
2. $a\tilde{b}, c\tilde{d}, \tilde{b}c, \tilde{d}a \in \mathbb{R}^m$.
3. $a\tilde{d} - b\tilde{c} = \pm 1$.

Since $y = (ax + b)(cx + d)^{-1} = ac^{-1} + (b - ac^{-1}d)(cx + d)^{-1}$, a conformal transformation can be decomposed as compositions of translation, dilation, reflection and inversion. This gives an *Iwasawa decomposition* for Möbius transformations. See [23] for more details.

The Dirac operator in \mathbb{R}^m is defined to be

$$D_x := \sum_{i=1}^m e_i \partial_{x_i}.$$

Note $D_x^2 = -\Delta_x$, where Δ_x is the Laplacian in \mathbb{R}^m . A \mathcal{Cl}_m -valued function $f(x)$ defined on a domain U in \mathbb{R}^m is left monogenic if $D_x f(x) = 0$. Since Clifford multiplication is not commutative in general, there is a similar definition for right monogenic functions. Sometimes, we will consider the Dirac operator D_u in a vector u rather than x .

In classical Clifford analysis, the k th order conformally invariant differential operator is D_x^k and a large number of results have been found, for instance, [17, 25, 29, 30]. In particular, the fundamental solutions and the intertwining operators for D_x^k are as follows.

Proposition 1. [25, 29] (**Fundamental solutions for D_x^k**)

Let $x \in \mathbb{R}^m$, the fundamental solutions $G_k(x)$ for D_x^k are as follows. When m is odd,

$$G_k(x) := \begin{cases} c_{2n} \|x\|^{2n-m}, & \text{if } k = 2n, n = 1, 2, \dots, \\ c_{2n-1} \frac{x}{\|x\|^{m-2n+2}}, & \text{if } k = 2n-1, n = 1, 2, \dots. \end{cases}$$

When m is even,

$$G_k(x) := \begin{cases} \frac{1}{\|x\|^{m-2n}}, & \text{if } k = 2n, n = 1, 2, \dots, \frac{m}{2} - 1, \\ \frac{x}{\|x\|^{m-2n+2}}, & \text{if } k = 2n-1, n = 1, 2, \dots, \frac{m}{2} - 1. \end{cases}$$

Proposition 2. [25, 29] (*Intertwining operators for D_x^k*)

Let $y = \varphi(x) = (ax + b)(cx + d)^{-1}$ be a Möbius transformation. Then we have

$$J_{-k}(\varphi, x) D_y^k f(y) = D_x^k J_k(\varphi, x) f((ax + b)(cx + d)^{-1}),$$

where

$$\begin{aligned} J_k(\varphi, x) &= \frac{\widetilde{cx + d}}{\|cx + d\|^{m-2j+2}}, \text{ if } k = 2j - 1, \\ J_k(\varphi, x) &= \|cy + d\|^{2j-m}, \text{ if } k = 2j; \\ J_{-k}(\varphi, x) &= \frac{cx + d}{\|cx + d\|^{m+2j}}, \text{ if } k = 2j - 1, \\ J_{-k}(\varphi, x) &= \|cx + d\|^{-m-2j}, \text{ if } k = 2j, \end{aligned}$$

and j is a positive integer. J_{-k} , J_k are called the intertwining operators and J_k is called the conformal weight for D_x^k .

In this paper, we will generalize k th ($k > 2$) order conformally invariant differential operators from classical Clifford analysis to higher spin theory as well as their fundamental solutions and intertwining operators. We start with introducing two well known polynomial spaces and the first and second order conformally invariant differential operators in higher spin theory as follows.

Let \mathcal{M}_k denote the space of $\mathcal{C}l_m$ -valued monogenic polynomials homogeneous of degree k . Note that if $h_k \in \mathcal{H}_k$, the space of $\mathcal{C}l_m$ -valued harmonic polynomials homogeneous of degree k , then $D_u h_k \in \mathcal{M}_{k-1}$, but $D_u u p_{k-1}(u) = (-m - 2k + 2)p_{k-1}(u)$, so

$$\mathcal{H}_k = \mathcal{M}_k \oplus u\mathcal{M}_{k-1}, \quad h_j = p_k + u p_{k-1}.$$

This is an *Almansi-Fischer decomposition* of \mathcal{H}_k [13]. In this Almansi-Fischer decomposition, we define P_k as the projection map

$$P_k : \mathcal{H}_k \longrightarrow \mathcal{M}_k.$$

Suppose U is a domain in \mathbb{R}^m . Consider a differentiable function $f : U \times \mathbb{R}^m \longrightarrow \mathcal{C}l_m$ such that, for each $x \in U$, $f(x, u)$ is a left monogenic polynomial homogeneous of degree k in u . Then the first order conformally invariant differential operator in higher spin theory, named as Rarita-Schwinger operator [6, 13], is defined by

$$R_k f(x, u) := P_k D_x f(x, u) = \left(\frac{u D_u}{m + 2k - 2} + 1 \right) D_x f(x, u). \quad (1)$$

Let $Z_k(u, v)$ be the reproducing kernel for \mathcal{M}_k , which satisfies

$$f(v) = \int_{\mathbb{S}^{m-1}} \overline{Z_k(u, v)} f(u) dS(u), \text{ for all } f(v) \in \mathcal{M}_k.$$

Then the fundamental solution for R_k is

$$E_{k,1}(x, u, v) = \frac{1}{\omega_{m-1}c_{k,1}} \frac{x}{||x||^m} Z_k\left(\frac{xux}{||x||^2}, v\right),$$

where constant $c_{k,1}$ is $\frac{m-2}{m+2k-2}$ and ω_{m-1} is the area of $(m-1)$ -dimensional unit sphere.

In other words, R_k can be considered as the inverse of $E_{k,1}(x, u, v)$ in the following sense.

Proposition 3. *For any $\phi(y, v) \in C^\infty(\mathbb{R}^m, \mathcal{M}_k)$ with compact support with respect to variable x , we have*

$$\iint_{\mathbb{R}^m} (R_k E_{k,1}(x - y, u, v), \phi(x, v))_v dx^m = \phi(y, u).$$

where

$$(f(v), g(v))_v = \int_{\mathbb{S}^{m-1}} f(v)g(v)dS(v)$$

is the Fischer-inner product for two Clifford valued polynomials.

The second order conformally invariant differential operator in higher spin theory, named the higher spin Laplace operator [3], is defined by

$$\mathcal{D}_2 = \Delta_x - \frac{4\langle u, D_x \rangle \langle D_u, D_x \rangle}{m+2k-2} + \frac{||u||^2 \langle D_u, D_x \rangle^2}{(m+2k-2)(m+2k-4)}.$$

Its fundamental solution is given by

$$E_{k,2}(x, u, v) = \frac{(m+2k-4)\Gamma(\frac{m}{2}-1)}{4(4-m)\pi^{\frac{m}{2}}} ||x||^{2-m} Z_k\left(\frac{xux}{||x||^2}, v\right),$$

where $Z_k(u, v)$ is the reproducing kernel for \mathcal{H}_k and satisfies

$$f(v) = \int_{\mathbb{S}^{m-1}} \overline{Z_k(u, v)} f(u) dS(u), \text{ for all } f(v) \in \mathcal{H}_k.$$

Also \mathcal{D}_2 can also considered as the inverse of $E_{k,2}(x, u, v)$ in a similar sense as for R_k . This will be studied in a more general setting in Section 4.1.

Though we have presented the Almansi-Fischer decomposition, the Dirac operator, and the Rarita-Schwinger operator here in terms of functions taking values in the real Clifford algebra \mathcal{Cl}_m , they can all be realized in the same way for spinor-valued functions in the complex Clifford algebra $\mathcal{Cl}_m(\mathbb{C})$, see [8]; we discuss spinors in the next section.

2.2 Irreducible representations of the Spin group

We now introduce three representations of $Spin(m)$. The first representation of the Spin group is used as the target space in spinor-valued theory and the other two representations of the Spin group are frequently used as target spaces in higher spin theory.

2.2.1 Spinor representation space \mathcal{S}

The most commonly used representation of the Spin group in $\mathcal{Cl}_m(\mathbb{C})$ -valued function theory is the spinor space. To this end, consider the complex Clifford algebra $\mathcal{Cl}_m(\mathbb{C})$ with even dimension $m = 2n$. The space of vectors \mathbb{C}^m is embedded in $\mathcal{Cl}_m(\mathbb{C})$ as

$$(x_1, x_2, \dots, x_m) \mapsto \sum_{j=1}^m x_j e_j : \mathbb{C}^m \hookrightarrow \mathcal{Cl}_m(\mathbb{C}).$$

We denote x for a *vector* in both interpretations. The *Witt basis* elements of \mathbb{C}^m are defined by

$$f_j := \frac{e_{2j-1} - ie_{2j}}{2}, \quad f_j^\dagger := -\frac{e_{2j-1} + ie_{2j}}{2}, \quad 1 \leq j \leq n.$$

Let $I := f_1 f_1^\dagger \dots f_n f_n^\dagger$. The space of *Dirac spinors* is defined as

$$\mathcal{S} := \mathcal{Cl}_m(\mathbb{C})I.$$

This is a representation of $Spin(m)$ under the following action

$$\rho(s)\mathcal{S} := s\mathcal{S}, \text{ for } s \in Spin(m).$$

Note \mathcal{S} is a left ideal of $\mathcal{Cl}_m(\mathbb{C})$. For more details, see [8]. An alternative construction of spinor spaces is given in the classic paper of Atiyah, Bott and Shapiro [2].

2.2.2 Homogeneous harmonic polynomials on $\mathcal{H}_k(\mathbb{R}^m, \mathbb{C})$

It is well known the space of harmonic polynomials is invariant under action of $Spin(m)$, since the Laplacian Δ_m is an $SO(m)$ invariant operator. It is not irreducible for $Spin(m)$, however, and can be decomposed into the infinite sum of k -homogeneous harmonic polynomials, $0 \leq k < \infty$. Each of these spaces is irreducible for $Spin(m)$. This brings the most familiar representations of $Spin(m)$: spaces of complex-valued k -homogeneous harmonic polynomials on \mathbb{R}^m , denoted by $\mathcal{H}_k := \mathcal{H}_k(\mathbb{R}^m, \mathbb{C})$. Since \mathbb{C} is considered as a scalar subspace of $\mathcal{Cl}_m(\mathbb{C})$, \mathcal{H}_k is also called a scalar-valued k -homogeneous harmonic polynomial spaces. The following action has been shown to be an irreducible representation of $Spin(m)$ [19, 21]:

$$\rho : Spin(m) \longrightarrow Aut(\mathcal{H}_k), \quad s \longmapsto (f(x) \mapsto \tilde{s}f(sx\tilde{s})s).$$

This can also be realized as follows

$$\begin{aligned} Spin(m) &\xrightarrow{\theta} SO(m) \xrightarrow{\rho} Aut(\mathcal{H}_k); \\ a &\longmapsto O_a \longmapsto (f(x) \mapsto f(O_a x)), \end{aligned}$$

where θ is the double covering map and ρ is the standard action of $SO(m)$ on a function $f(x) \in \mathcal{H}_k$ with $x \in \mathbb{R}^m$.

2.2.3 Homogeneous monogenic polynomials on \mathcal{Cl}_m

In \mathcal{Cl}_m -valued function theory, the previously mentioned Almansi-Fischer decomposition shows we can also decompose the space of k -homogeneous harmonic polynomials:

$$\mathcal{H}_k = \mathcal{M}_k \oplus u\mathcal{M}_{k-1}.$$

If we restrict \mathcal{M}_k to the spinor valued subspace, we have another important representation of $Spin(m)$: the space of k -homogeneous spinor-valued monogenic polynomials on \mathbb{R}^m , henceforth denoted by $\mathcal{M}_k := \mathcal{M}_k(\mathbb{R}, \mathcal{S})$. Specifically, the following action has been shown to be an irreducible representation of $Spin(m)$ [19, 21]:

$$\pi : Spin(m) \longrightarrow Aut(\mathcal{M}_k), \quad s \longmapsto f(x) \mapsto \tilde{s}f(sx\tilde{s}).$$

3 Stein-Weiss type operators

In classical Clifford analysis, the Euclidean Dirac operator was initially motivated from Stokes' Theorem [28] and Clifford algebras were used to study it. When we consider function theory in higher spin spaces, since these functions take values in irreducible representations of the Spin group, it turns out representation theory provides a quite different approach for operator theory in higher spin spaces. Abundant results have been found with this approach: for instance, [3, 5, 15, 16]. In 1968, Stein and Weiss [32] pointed out that many first-order differential operators can be constructed as projections of generalized gradients with the help of representation theory. Fegan [17] showed that such operators are conformally invariant with certain conditions. In [9, 33], the Euclidean Dirac and Rarita-Schwinger operators were constructed as Stein-Weiss type operators. Since this construction generalizes further to representations of principal bundles over oriented Riemannian spin manifolds, by which one constructs the Atiyah-Singer Dirac operator, we argue the Stein and Weiss construction is the natural way to construct other Dirac type operators as in [19, 33]. In this section, we recall the constructions of the Euclidean Dirac and Rarita-Schwinger operators as Stein-Weiss type operators from [9]. Motivated by this representation theoretic approach, we will construct other higher order conformally invariant differential operators in higher spin spaces in the next section.

Assume U is a finite dimensional inner product complex vector space, V is a m -dimensional inner product complex vector space. Denote the groups of all automorphisms of U and V by $GL(U)$ and $GL(V)$, respectively. Suppose $\rho_1 : G \rightarrow GL(U)$ and $\rho_2 : G \rightarrow GL(V)$ are irreducible representations of a compact Lie group G . Let $f(x)$ be a differentiable function defined on a domain $\Omega \subset \mathbb{R}^m$ with values in U . We wish to define the gradient $\nabla f(x)$ as a function from the same domain Ω but with values in $U \otimes V$. Suppose that $\{\zeta_\alpha\}$ is an orthonormal basis in U and $f(x) = \sum_\alpha f_\alpha(x)\zeta_\alpha$. Let $\{e_1, \dots, e_m\}$ be the standard basis of V arising from the identification of V with \mathbb{C}^m . Then a basis (over \mathbb{C}) of $U \otimes V$ is $\{\zeta_\alpha \otimes e_i\}_{\alpha,i}$ and

$$\nabla f(x) = \sum_{\alpha,i} \frac{\partial f_\alpha(x)}{\partial x_i} \zeta_\alpha \otimes e_i.$$

In this paper, we rewrite $\nabla f(x)$ as follows for convenience,

$$\nabla f(x) = \sum_i \frac{\partial f(x)}{\partial x_i} e_i.$$

Since $U \otimes V$ is not necessarily irreducible as a tensor product representation of G , we denote by $U[\times]V$ the irreducible subrepresentation of $U \otimes V$ whose representation space has largest dimension. This is known as the Cartan product of ρ_1 and ρ_2 . For more details, see [14, 32]. Using the inner products on U and V , we can write

$$U \otimes V = (U[\times]V) \oplus (U[\times]V)^\perp.$$

If we denote by E and E^\perp the orthogonal projections onto $U[\times]V$ and $(U[\times]V)^\perp$, respectively, then we define differential operators D and D^\perp associated to ρ_1 and ρ_2 by

$$D = E\nabla \text{ and } D^\perp = E^\perp\nabla.$$

These are named *Stein-Weiss type operators* after [32]. The importance of this construction is that one can reconstruct many first-order differential operators with it by choosing proper representation spaces U and V for a Lie group G , such as the Euclidean Dirac operators [32, 33] and Rarita-Schwinger operators [19] that we now proceed to discuss.

1. Dirac operators

Here we only show the odd dimension case, but the even dimension case is similar.

Theorem 1. *Let ρ_1 be the representation of the spin group given by the standard representation of $SO(m)$ on \mathbb{R}^m*

$$\rho_1 : Spin(m) \rightarrow SO(m) \rightarrow GL(\mathbb{R}^m)$$

and let ρ_2 be the spin representation on the spinor space \mathcal{S} . Then the Euclidean Dirac operator is the differential operator given by projecting the gradient onto $(\mathbb{R}^m[\times]\mathcal{S})^\perp$ when $m = 2n + 1$.

Outline proof: The proof is exactly that appearing in [32]. Let $\{e_1, \dots, e_m\}$ be an orthonormal basis of \mathbb{R}^m and $x = (x_1, \dots, x_m) \in \mathbb{R}^m$. For a function $f(x)$ having values in \mathcal{S} , we must show that the system

$$\sum_{i=1}^m e_i \frac{\partial f}{\partial x_i} = 0$$

is equivalent to the system

$$D^\perp f = E^\perp \nabla f = 0.$$

We have

$$\mathbb{R}^m \otimes \mathcal{S} = \mathbb{R}^m[\times] \mathcal{S} \oplus (\mathbb{R}^m[\times] \mathcal{S})^\perp$$

and [32] provides an embedding map

$$\begin{aligned} \eta : \mathcal{S} &\hookrightarrow \mathbb{R}^m \otimes \mathcal{S}, \\ \omega &\mapsto \frac{1}{\sqrt{m}}(e_1\omega, \dots, e_m\omega). \end{aligned}$$

Indeed, this embedding is an isomorphism from \mathcal{S} into $\mathbb{R}^m \otimes \mathcal{S}$. For the proof, we refer the reader to *page 175* of [32]. Thus, we have

$$\mathbb{R}^m \otimes \mathcal{S} = \mathbb{R}^m[\times] \mathcal{S} \oplus \eta(\mathcal{S}).$$

Consider the equation $D^\perp f = E^\perp \nabla f = 0$, where f has values in \mathcal{S} . So ∇f has values in $\mathbb{R}^m \otimes \mathcal{S}$, and the condition $D^\perp f = 0$ is equivalent to ∇f being orthogonal to $\eta(\mathcal{S})$. This is precisely the statement that

$$\sum_{i=1}^m \left(\frac{\partial f}{\partial x_i}, e_i \omega \right) = 0, \quad \forall \omega \in \mathcal{S}.$$

Notice, however, that as an endomorphism of $\mathbb{R}^m \otimes \mathcal{S}$, we have $-e_i$ as the dual of e_i . Hence the equation above becomes

$$\sum_{i=1}^m \left(e_i \frac{\partial f}{\partial x_i}, \omega \right) = 0, \quad \forall \omega \in \mathcal{S},$$

which says precisely that f must be in the kernel of the Euclidean Dirac operator. This completes the proof. \square

2. Rarita-Schwinger operators

Theorem 2. *Let ρ_1 be defined as above and ρ_2 is the representation of $Spin(m)$ on \mathcal{M}_k . Then as a representation of $Spin(m)$, we have the following decomposition*

$$\mathcal{M}_k \otimes \mathbb{R}^m \cong \mathcal{M}_k[\times] \mathbb{R}^m \oplus \mathcal{M}_k \oplus \mathcal{M}_{k-1} \oplus \mathcal{M}_{k,1},$$

where $\mathcal{M}_{k,1}$ is a simplicial monogenic polynomial space as a $Spin(m)$ representation (see more details in [3]). The Rarita-Schwinger operator is the differential operator given by projecting the gradient onto the \mathcal{M}_k component.

Proof. Consider $f(x, u) \in C^\infty(\mathbb{R}^m, \mathcal{M}_k)$. We observe that the gradient of $f(x, u)$ satisfies

$$\nabla f(x, u) = (\partial_{x_1}, \dots, \partial_{x_m})f(x, u) = (\partial_{x_1}f(x, u), \dots, \partial_{x_m}f(x, u)) \in \mathcal{M}_k \otimes \mathbb{R}^m.$$

A similar argument as in *page 181* of [32] shows

$$\mathcal{M}_k \otimes \mathbb{R}^m = \mathcal{M}_k[\times]\mathbb{R}^m \oplus V_1 \oplus V_2 \oplus V_3,$$

where $V_1 \cong \mathcal{M}_k$, $V_2 \cong \mathcal{M}_{k-1}$ and $V_3 \cong \mathcal{M}_{k,1}$ as $Spin(m)$ representations. Similar arguments as on *page 175* of [32] show

$$\theta : \mathcal{M}_k \longrightarrow \mathcal{M}_k \otimes \mathbb{R}^m, \quad q_k(u) \mapsto (q_k(u)e_1, \dots, q_k(u)e_m)$$

is an isomorphism from \mathcal{M}_k into $\mathcal{M}_k \otimes \mathbb{R}^m$. Hence, we have

$$\mathcal{M}_k \otimes \mathbb{R}^m = \mathcal{M}_k[\times]\mathbb{R}^m \oplus \theta(\mathcal{M}_k) \oplus V_2 \oplus V_3.$$

Let P'_k be the projection map from $\mathcal{M}_k \otimes \mathbb{R}^m$ to $\theta(\mathcal{M}_k)$. Consider the equation $P'_k \nabla f(x, u) = 0$ for $f(x, u) \in C^\infty(\mathbb{R}^m, \mathcal{M}_k)$. Then, for each fixed x , $\nabla f(x, u) \in \mathcal{M}_k \otimes \mathbb{R}^m$ and the condition $P'_k \nabla f(x, u) = 0$ is equivalent to ∇f being orthogonal to $\theta(\mathcal{M}_k)$. This says precisely

$$\sum_{i=1}^m (q_k(u)e_i, \partial_{x_i}f(x, u))_u = 0, \quad \forall q_k(u) \in \mathcal{M}_k,$$

where $(p(u), q(u))_u = \int_{\mathbb{S}^{m-1}} \overline{p(u)}q(u)dS(u)$ is the Fischer inner product for any pair of \mathcal{Cl}_m -valued polynomials. Since $-e_i$ is the dual of e_i as an endomorphism of $\mathcal{M}_k \otimes \mathbb{R}^m$, the previous equation becomes

$$\sum_{i=1}^m (q_k(u), e_i \partial_{x_i}f(x, u)) = (q_k(u), D_x f(x, u))_u = 0.$$

Since $f(x, u) \in \mathcal{M}_k$ for fixed x , then $D_x f(x, u) \in \mathcal{H}_k$. According to the Almansi-Fischer decomposition, we have

$$D_x f(x, u) = f_1(x, u) + u f_2(x, u), \quad f_1(x, u) \in \mathcal{M}_k \text{ and } f_2(x, u) \in \mathcal{M}_{k-1}.$$

We then obtain $(q_k(u), f_1(x, u))_u + (q_k(u), u f_2(x, u))_u = 0$. However, the Clifford-Cauchy theorem [13] shows $(q_k(u), u f_2(x, u))_u = 0$. Thus, the equation $P'_k \nabla f(x, u) = 0$ is equivalent to

$$(q_k(u), f_1(x, u))_u = 0, \quad \forall q_k(u) \in \mathcal{M}_k.$$

Hence, $f_1(x, u) = 0$. We also know, from the construction of the Rarita-Schwinger operator (see (1)), that $f_1(x, u) = R_k f(x, u)$. Therefore, the Stein-Weiss type operator $P'_k \nabla$ is precisely the Rarita-Schwinger operator in this context. \square

We have demonstrated one application of the Representation-Theoretic approach to Clifford analysis: the Stein-Weiss generalized gradient construction for the Euclidean Dirac and Rarita-Schwinger operators. The operators are realized on irreducible representations of the Spin group. In higher spin theory, we consider operators on functions taking values in irreducible spin representations that have higher spin, i.e., \mathcal{H}_k or \mathcal{M}_k . Seeing our success already, we now use the Representation-Theoretic approach to extend the higher spin theory to arbitrary order conformally invariant differential operators of arbitrary spin.

4 Construction and conformal invariance

Denote the arbitrary t -th-order conformally invariant differential operator

$$\mathcal{D}_t : C^\infty(\mathbb{R}^m, V) \longrightarrow C^\infty(\mathbb{R}^m, V),$$

where the target space V is \mathcal{M}_k or \mathcal{H}_k . Thanks to results in [34, 35], the existence and uniqueness (up to a multiplicative constant) of \mathcal{D}_t are already established. More specifically, even order conformally invariant differential operators only exist when $V = \mathcal{H}_k$ and odd order conformally invariant differential operators only exist when $V = \mathcal{M}_k$. This can be easily obtained by taking \mathcal{M}_k or \mathcal{H}_k as the irreducible representation of $Spin(m)$ in Theorems 2 and 3 in [35]; these theorems also give the conformal weights of \mathcal{D}_t , which provide the intertwining operators of \mathcal{D}_t . More specifically, the following result can be obtained from [35].

Proposition 4. *Suppose $y \in \mathbb{R}^m$, $y' = (ay + b)(cy + d)^{-1}$ is a Möbius transformation and $u' = \frac{(cy + d)\widetilde{u(cy + d)}}{\|cy + d\|^2}$. Then*

$$\mathcal{D}_{t,y',u'} = J_{-t}^{-1}(\varphi, y) \mathcal{D}_{t,y,u} J_t(\varphi, y), \quad (2)$$

where

$$\begin{aligned} J_t(\varphi, y) &= \frac{\widetilde{cy + d}}{\|cy + d\|^{m-2j+2}}, \text{ if } t = 2j - 1, \\ J_t(\varphi, y) &= \|cy + d\|^{2j-m}, \text{ if } t = 2j; \\ J_{-t}(\varphi, y) &= \frac{cy + d}{\|cy + d\|^{m+2j}}, \text{ if } t = 2j - 1, \\ J_{-t}(\varphi, y) &= \|cy + d\|^{-m-2j}, \text{ if } t = 2j, \end{aligned}$$

and j is a positive integer. Here J_t is called the conformal weight for \mathcal{D}_t , and J_t and J_{-t} are called the intertwining operators for \mathcal{D}_t .

More details can be found in [11] Section 3.2. Let $Z_k(u, v)$ be the reproducing kernel of \mathcal{M}_k , which satisfies

$$f(v) = \int_{\mathbb{S}^{m-1}} \overline{Z_k(u, v)} f(u) dS(u), \text{ for all } f(v) \in \mathcal{M}_k.$$

Recall the fundamental solution of the Rarita-Schwinger operator is $c \frac{x}{||x||^m} Z_k(\frac{xux}{||x||^2}, v)$, where c is a non-zero constant [6]. We call $\frac{x}{||x||^m}$ the conformal weight factor and $Z_k(\frac{xux}{||x||^2}, v)$ the reproducing kernel factor. The fundamental solution of D_x^k is [25]

$$c_{2j+1} \frac{x}{||x||^{m-2j}}, \text{ if } k = 2j + 1, \text{ and } c_{2j} ||x||^{2j-m}, \text{ if } k = 2j,$$

where c_{2j+1} and c_{2j} are non-zero constants. However, when dimension m is even, we also require that $k < m$, because for instance, when $m = k = 2j$, the only candidate of fundamental solution is a constant. We expect the fundamental solutions of our higher order higher spin conformally invariant differential operators \mathcal{D}_t to factor into two parts: a conformal weight factor and a reproducing kernel factor, behaving as follows.

1. The conformal weight factor, i.e., $\frac{x}{||x||^{m-2j}}$ or $||x||^{2j-m}$ term, changes with increasing order similar to the conformal weight for powers of the Dirac operator, differing in the even and odd cases.
2. The reproducing kernel factor, i.e., $Z_k(\frac{xux}{||x||^2}, v)$ term, changes with increasing degree of homogeneity of the target polynomial space similar to the Rarita-Schwinger operator, differing according to whether it is the space of harmonic or monogenic polynomials.

Thus we guess candidates for the fundamental solutions as follows.

1. For \mathcal{D}_{2j} , $c ||x||^{2j-m} Z_k(\frac{xux}{||x||^2}, v)$, where $Z_k(u, v)$ is the reproducing kernel of \mathcal{H}_k .
2. For \mathcal{D}_{2j-1} , $c \frac{x}{||x||^{m-2j+2}} Z_k(\frac{xux}{||x||^2}, v)$, where $Z_k(u, v)$ is the reproducing kernel of \mathcal{M}_k .

With similar arguments as in [6, 11], we have the following result.

Proposition 5. *The fundamental solution for \mathcal{D}_{2j} is $c_{2j} ||x||^{2j-m} Z_k(\frac{xux}{||x||^2}, v)$, where $Z_k(u, v)$ is the reproducing kernel of \mathcal{H}_k and c_{2j} is a non-zero constant. The fundamental solution for \mathcal{D}_{2j-1} is $c_{2j-1} \frac{x}{||x||^{m-2j+2}} Z_k(\frac{xux}{||x||^2}, v)$, where $Z_k(u, v)$ is the reproducing kernel of \mathcal{M}_k and c_{2j-1} is a non-zero constant.*

Proof. We only give the proof for the fundamental solutions of \mathcal{D}_{2j} . A similar argument also applies for \mathcal{D}_{2j-1} . Let $Z_k(u, v)$ be the reproducing kernel of \mathcal{H}_k , which can be considered as the identity of $End(\mathcal{H}_k)$ and satisfies

$$P_k(v) = (Z_k(u, v), P_k(u))_u = \int_{S^{m-1}} \overline{Z_k(u, v)} P_k(u) dS(u), \text{ for any } P_k(u) \in \mathcal{H}_k.$$

A homogeneous $End(\mathcal{H}_k)$ -valued C^∞ -function $x \rightarrow E(x)$ on $\mathbb{R}^m \setminus \{0\}$ satisfying $\mathcal{D}_{2j}E(x) = \delta(x)Z_k(u, v)$ is referred to as a fundamental solution for the operator \mathcal{D}_{2j} . We will show that such a fundamental solution has the form $E_{k,2j}(x, u, v) = c_{2j}||x||^{2j-m}Z_k(\frac{xux}{||x||^2}, v)$.

Since $Z_k(u, v)$ is a trivial solution of \mathcal{D}_{2j} , according to the invariance of \mathcal{D}_{2j} under inversion, we obtain a non-trivial solution $\mathcal{D}_{2j}E_{k,2j}(x, u, v) = 0$ in $\mathbb{R}^m \setminus \{0\}$, this can be easily verified from Proposition 4 when the Möbius transformation is inversion. Clearly the function $E_{k,2j}(x, u, v)$ is homogeneous of degree $2j - m$ in x , so $\mathcal{D}_{2j}E_{k,2j}(x, u, v)$ is homogeneous of degree $-m$ in x and it belongs to $L_1^{loc}(\mathbb{R}^m)$. Because $\delta(x)$ is the only (up to a multiple) distribution homogeneous of degree $-m$ with support at the origin, we have in the sense of distributions:

$$\mathcal{D}_{2j}E_{k,2j}(x, u, v) = \delta(x)P_k(u, v)$$

for some $P_k(u, v) \in \mathcal{H}_k \otimes \mathcal{H}_k^*$. Then we have

$$\begin{aligned} & \int_{S^{m-1}} \mathcal{D}_{2j} \overline{E_{k,2j}(x, u, v)} Q_k(v) dS(v) \\ &= \delta(x) \int_{S^{m-1}} \overline{P_k(u, v)} Q_k(v) dS(v). \end{aligned}$$

Now, for all $Q_k \in \mathcal{H}_k$, we have

$$\begin{aligned} & \int_{S^{m-1}} \mathcal{D}_{2j} \overline{E_{k,2j}(x, u, v)} Q_k(v) dS(v) \\ &= \mathcal{D}_{2j} \int_{S^{m-1}} c_{2j} ||x||^{2j-m} \overline{Z_k(\frac{xux}{||x||^2}, v)} Q_k(v) dS(v) \\ &= \mathcal{D}_{2j} \int_{S^{m-1}} c_{2j} ||x||^{2j-m} \overline{Z_k(\frac{xux}{||x||^2}, \frac{xv'x}{||x||^2})} Q_k(\frac{xv'x}{||x||^2}) dS(v'), \end{aligned} \tag{3}$$

where in the last line we made a change of variables in the second argument of Z_k . Since $Z_k(u, v)$ is invariant under reflection and $\frac{xux}{||x||^2}$ is a reflection of the variable u in the direction of x , in other words ([19]),

$$Z_k(u, v) = \frac{x}{||x||} Z_k(\frac{xux}{||x||^2}, \frac{xvx}{||x||^2}) \frac{x}{||x||} = -Z_k(\frac{xux}{||x||^2}, \frac{xvx}{||x||^2}).$$

The last equation comes from that $Z_k(\frac{xux}{||x||^2}, \frac{xvx}{||x||^2}) \in \mathcal{H}_k$, which is scalar valued. Hence, we can commute $\frac{xvx}{||x||^2}$ and $Z_k(\frac{xux}{||x||^2}, \frac{xvx}{||x||^2})$. Further, $x^2 = -||x||^2$.

Therefore, equation (3) becomes

$$\begin{aligned} & \mathcal{D}_{2j} \int_{\mathbb{S}^{m-1}} -c_{2j} \overline{Z_k(u, v')} ||x||^{2j-m} Q_k\left(\frac{xv'x}{||x||^2}\right) dS(v') \\ &= -c_{2j} \mathcal{D}_{k,2j} ||x||^{2j-m} Q_k\left(\frac{xux}{||x||^2}\right). \end{aligned}$$

Hence, we obtain

$$\delta(x) \int_{\mathbb{S}^{m-1}} \overline{P_k(u, v)} Q_k(v) dS(v) = -c_{2j} \mathcal{D}_{k,2j} ||x||^{2j-m} Q_k\left(\frac{xux}{||x||^2}\right).$$

As the reproducing kernel $Z_k(u, v)$ is invariant under the $Spin(m)$ -representation

$$H : f(u, v) \mapsto \tilde{s}f(su\tilde{s}, sv\tilde{s})s,$$

the kernel $E_{k,2j}(x, u, v)$ is also $Spin(m)$ -invariant:

$$\tilde{s}E_{k,2j}(sx\tilde{s}, su\tilde{s}, sv\tilde{s})s = E_{k,2j}(x, u, v).$$

From this it follows that $P_k(u, v)$ must be also invariant under H . Let now ϕ be a test function with $\phi(0) = 1$. Let L be the action of $Spin(m)$ given by $L : f(u) \mapsto \tilde{s}f(su\tilde{s})s$. Then

$$\begin{aligned} & \langle \mathcal{D}_{2j} \left(-c_{2j} ||x||^{2j-m} L\left(\frac{x}{||x||}\right) L(s) Q_k(u) \right), \phi(x) \rangle \\ &= \int_{\mathbb{S}^{m-1}} \overline{P_k(u, v)} L(s) Q_k(v) dS(v) \\ &= L(s) \int_{\mathbb{S}^{m-1}} \overline{P_k(u, v)} Q_k(v) dS(v) \\ &= \langle L(s) \left(-\mathcal{D}_{2j} c_{2j} ||x||^{2j-m} L\left(\frac{x}{||x||}\right) Q_k(u) \right), \phi(x) \rangle. \end{aligned}$$

In this way we have constructed an element of $End(\mathcal{H}_k)$ commuting with the L -representation of $Spin(m)$ that is irreducible; see Section 2.2.2. By Schur's Lemma ([18]) in representation theory, it follows that $P_k(u, v)$ must be the reproducing kernel $Z_k(u, v)$ if we choose c_{2j} properly. Hence

$$\mathcal{D}_{2j} E_{k,2j}(x, u, v) = \delta(x) Z_k(u, v).$$

□

We initially expect when the dimension m is even, we must restrict order $2j$ or $2j - 1$ to be less than m , analogous to the powers of the Dirac operator (see Proposition 2). However, the reproducing kernel factor, i.e., the $Z_k(\frac{xux}{||x||^2}, v)$ term in the fundamental solutions, renders this restriction on the order unnecessary for even dimensions. After we

can show these fundamental solutions are conformally invariant, constructing a conformally invariant differential operator becomes finding an operator which has a particular fundamental solution.

We already found the fundamental solutions for k th ($k \geq 1$) order conformally invariant differential operators. This provides us a simple way to recover the higher spin Laplace operator up to a multiplicative constant instead of using generalized symmetries as in [3]. Consider the twistor and dual twistor operators from the same reference:

$$\begin{aligned} T_{k,2} &= \langle u, D_x \rangle - \frac{||u||^2 \langle D_u, D_x \rangle}{m + 2k - 4} : C^\infty(\mathbb{R}^m, \mathcal{H}_{k-1}) \longrightarrow C^\infty(\mathbb{R}^m, \mathcal{H}_k), \\ T_{k,2}^* &= \langle D_u, D_x \rangle : C^\infty(\mathbb{R}^m, \mathcal{H}_k) \longrightarrow C^\infty(\mathbb{R}^m, \mathcal{H}_{k-1}). \end{aligned}$$

The second order operators Δ_x and $T_{k,2}T_{k,2}^*$ map from $C^\infty(\mathbb{R}^m, \mathcal{H}_k)$ to $C^\infty(\mathbb{R}^m, \mathcal{H}_k)$ and do not change the degree of homogeneity of the variable u ; more details can be found in [3]. These are scalar-valued as desired, since \mathcal{H}_k is a scalar-valued function space. It is reasonable, then, to guess the second order bosonic operator of spin k (the higher spin Laplace operator) is a linear combination of these two operators. By our earlier arguments, if there is a linear combination of Δ_x and $T_{k,2}T_{k,2}^*$ that annihilates $c||x||^{2-m}Z_k(\frac{xux}{||x||^2}, v)$, where $Z_k(u, v)$ is the reproducing kernel of \mathcal{H}_k and c is a non-zero constant, then that operator is the higher spin Laplace operator up to a multiplicative constant:

$$\mathcal{D}_2 = \Delta_x - \frac{4T_{k,2}T_{k,2}^*}{m + 2k - 2}.$$

In the rest of this paper, we first introduce convolution type operators associated to fundamental solutions, then we point out fundamental solutions are actually the inverses of the corresponding differential operators in the sense of previous type of convolution. Further we show these convolution type operators are conformally invariant. Therefore, operators with such fundamental solutions are also conformally invariant, considering they are the inverses of their fundamental solutions in the sense of convolution. This also brings us a class of conformally invariant convolution type operators; their inverses, when they exist, are conformally invariant pseudo-differential operators. In classical Clifford analysis, such convolution type operators can be recovered as Knapp-Stein intertwining operators with the help of spinor principal series representations of $Spin(m)$, see [7]. Hence, our convolution type operators should also be recovered as Knapp-Stein intertwining operators with the principal series representations induced by the polynomial representations of $Spin(m)$ defined in Section 2.2. However, this is not obvious, and it will be investigated in more detail in an upcoming paper.

Since the even and odd order conformally invariant differential operators have different target spaces, we will show the constructions in even and odd order cases separately. The even order operators, which have integer spin, are named bosonic operators in analogy with bosons in physics, which are particles of integer spin. Correspondingly, the odd order operators, which have half-integer spin, are named fermionic operators after fermions,

which are particles of half-integer spin. It is worth pointing out that the non-zero constants in the fundamental solutions of our conformally invariant differential operators are also determined here. This provides the undetermined constants of the fundamental solutions in the lower spin case in [11].

4.1 Convolution type operators

Assume $E_k(x, u, v)$ is the fundamental solution of \mathcal{D}_k . Then we define a convolution operator as follows.

$$\Phi(f)(y, v) = E_k(x - y, u, v) * f(x, u) := \int_{\mathbb{R}^m} \int_{\mathbb{S}^{m-1}} E_k(x - y, u, v) f(x, u) dS(u) dx^m$$

Notice this is not the usual convolution operator, as it has an integral over the unit sphere with respect to variable u . It is worth pointing out that these convolution type operators are actually examples of Knapp-Stein intertwining operators, see [7]. Since $E_k(x, u, v)$ is the fundamental solution of \mathcal{D}_k , we have

$$\mathcal{D}_{k,x,u} E_k(x - y, u, v) * f(x, u) := \int_{\mathbb{R}^m} \int_{\mathbb{S}^{m-1}} \mathcal{D}_{k,x,u} E_k(x - y, u, v) f(x, u) dS(u) dx^m = f(y, v),$$

where $f(y, v) \in C^\infty(\mathbb{R}^m, U)$ ($U = \mathcal{H}_k$ or \mathcal{M}_k) with compact support in y for each $v \in \mathbb{R}^m$. Hence, we have $\mathcal{D}_k E_k = Id$ and $E_k^{-1} = \mathcal{D}_k$ in the sense above. This implies that if we can show our convolution operator Φ is conformally invariant, then its corresponding differential operator is also conformally invariant by taking its inverse.

Denote

$$E_{2j}(x, u, v) = ||x||^{2j-m} Z_k\left(\frac{xux}{||x||^2}, v\right) \text{ and } E_{2j-1}(x, u, v) = \frac{x}{||x||^{m-2j+2}} Z_k\left(\frac{xux}{||x||^2}, v\right),$$

where $Z_k(u, v)$ is the reproducing kernel of \mathcal{H}_k in the even case and the reproducing kernel of \mathcal{M}_k in the odd case.

Next we will show the above convolution operator Φ is conformally invariant under Möbius transformations. Thanks to the Iwasawa decomposition, it suffices to verify it is conformally invariant under orthogonal transformation, inversion, translation, and dilation. Conformal invariance under translation and dilation is trivial; hence, we only show the orthogonal transformation and inversion cases here. Incidentally, this method of proof is the first method we mentioned in the introduction for constructing conformally invariant operators, expect for a convolution operator rather than a differential operator; such a method was also mentioned, but not used, in [10, 11].

Proposition 6. (Orthogonal transformation) Suppose $a \in Spin(m)$ and $x \in \mathbb{R}^m$. If $x' = ax\tilde{a}$, $y' = ay\tilde{a}$, $u' = au\tilde{a}$, and $v' = av\tilde{a}$, then

1. $E_{2j}(x' - y', u', v') * f(x', u') = E_k(x - y, u, v) * f(ax\tilde{a}, au\tilde{a})$,
2. $E_{2j-1}(x' - y', u', v') * f(x', u') = aE_{2j-1}(x - y, u, v)\tilde{a} * f(ax\tilde{a}, au\tilde{a})$

Proof. Case 1. Let $f(x, u) \in C^\infty(\mathbb{R}^m, \mathcal{H}_k)$. Since the reproducing kernel of \mathcal{H}_k is rotationally invariant, $ax\tilde{a}$ is a rotation of x in the direction of a for $a \in Spin(m)$, and $a\tilde{a} = 1$, we have

$$\begin{aligned}
& E_{2j}(x' - y', u', v') * f(x', u') \\
&= \int_{\mathbb{R}^m} \int_{\mathbb{S}^{m-1}} \|x' - y'\|^{2j-m} Z_k\left(\frac{(x' - y')u'(x' - y')}{\|x' - y'\|^2}, v'\right) f(x', u') dS(u') dx'^m \\
&= \int_{\mathbb{R}^m} \int_{\mathbb{S}^{m-1}} \|a(x - y)\tilde{a}\|^{2j-m} Z_k\left(\frac{a(x - y)\tilde{a}au\tilde{a}a(x - y)\tilde{a}}{\|a(x - y)\tilde{a}\|^2}, av\tilde{a}\right) f(ax\tilde{a}, au\tilde{a}) dS(u) dx^m \\
&= \int_{\mathbb{R}^m} \int_{\mathbb{S}^{m-1}} \|x - y\|^{2j-m} Z_k\left(\frac{a(x - y)u(x - y)\tilde{a}}{\|x - y\|^2}, av\tilde{a}\right) f(ax\tilde{a}, au\tilde{a}) dS(u) dx^m \\
&= \int_{\mathbb{R}^m} \int_{\mathbb{S}^{m-1}} \|x - y\|^{2j-m} Z_k\left(\frac{(x - y)u(x - y)}{\|x - y\|^2}, v\right) f(ax\tilde{a}, au\tilde{a}) dS(u) dx^m \\
&= E_{2j}(x - y, u, v) * f(ax\tilde{a}, au\tilde{a})
\end{aligned}$$

Case 2. Since the reproducing kernel of \mathcal{M}_k has the property

$$Z_k(u, v) = \tilde{a}Z_k(au\tilde{a}, av\tilde{a})a$$

for $a \in Spin(m)$, similar argument as in *Case 1* gives the result. \square

Proposition 7. (Inversion) Suppose $x \in \mathbb{R}^m$. If $x' = x^{-1} = -\frac{x}{\|x\|^2}$, $y' = y^{-1} = -\frac{y}{\|y\|^2}$, $u' = \frac{yuy}{\|y\|^2}$ and $v' = \frac{xvx}{\|x\|^2}$, then

1. $E_{2j}(x' - y', u', v') * f(x', u') = -\|y\|^{m-2j} E_{2j}(x - y, u, v) \|x\|^{-m-2j} * f(x^{-1}, \frac{yuy}{\|y\|^2})$,
2. $E_{2j-1}(x' - y', u', v') * f(x', u') = -\left(\frac{y}{\|y\|^{m-2j+2}}\right)^{-1} E_{2j-1}(x - y, u, v) \frac{x}{\|x\|^{m-2j}} * f(x^{-1}, \frac{yuy}{\|y\|^2})$.

Proof. Case 1. Suppose $f(x, u) \in C^\infty(\mathbb{R}^m, \mathcal{H}_k)$. Notice

$$x^{-1} - y^{-1} = -x^{-1}(x - y)y^{-1} = -y^{-1}(x - y)x^{-1} = -\frac{x}{\|x\|^2}(x - y)\frac{y}{\|y\|^2} = -\frac{y}{\|y\|^2}(x - y)\frac{x}{\|x\|^2}.$$

Recall that, as the reproducing kernel of \mathcal{H}_k , $Z_k(u, v)$ has the property

$$Z_k(u, v) = -Z_k\left(\frac{xux}{\|x\|^2}, \frac{xvx}{\|x\|^2}\right)$$

for $x \in \mathbb{R}^m$. Hence, we have

$$\begin{aligned}
& E_{2j}(x' - y', u', v') * f(x', u') \\
&= \int_{\mathbb{R}^m} \int_{\mathbb{S}^{m-1}} \|x' - y'\|^{2j-m} Z_k\left(\frac{(x' - y')u'(x' - y')}{\|x' - y'\|^2}, v'\right) f(x', u') dS(u') dx'^m \\
&= \int_{\mathbb{R}^m} \int_{\mathbb{S}^{m-1}} \|x^{-1}(x - y)y^{-1}\|^{2j-m} Z_k\left(\frac{x(x - y)yuy(x - y)x}{\|y\|(x - y)x\|^2}, v'\right) f(x^{-1}, u') j(x^{-1}) dS(u') dx^m \\
&= \int_{\mathbb{R}^m} \int_{\mathbb{S}^{m-1}} -\|x^{-1}(x - y)y^{-1}\|^{2j-m} Z_k\left(\frac{(x - y)u(x - y)}{\|x - y\|^2}, v\right) f(x^{-1}, \frac{yuy}{\|y\|^2}) j(x^{-1}) dS(u) dx^m
\end{aligned}$$

where $j(x^{-1}) = ||x||^{-2m}$ is the Jacobian. Hence,

$$\begin{aligned}
&= - \int_{\mathbb{R}^m} \int_{\mathbb{S}^{m-1}} ||y||^{m-2j} ||x-y||^{2j-m} Z_k\left(\frac{(x-y)u(x-y)}{||x-y||^2}, v\right) ||x||^{-m-2j} f(x^{-1}, \frac{yuy}{||y||^2}) dS(u) dx^m \\
&= - ||y||^{m-2j} E_{2j}(x-y, u, v) ||x||^{-m-2j} * f(x^{-1}, \frac{yuy}{||y||^2}).
\end{aligned}$$

Case 2. Recall that, as the reproducing kernel of \mathcal{M}_k , $Z_k(u, v)$ has the property

$$Z_k(u, v) = -\frac{x}{||x||} Z_k\left(\frac{xux}{||x||^2}, \frac{xvx}{||x||^2}\right) \frac{x}{||x||}$$

for $x \in \mathbb{R}^m$. Then, by arguments similar to those above, we have

$$\begin{aligned}
&E_{2j-1}(x' - y', u', v') * f(x', u') \\
&= \int_{\mathbb{R}^m} \int_{\mathbb{S}^{m-1}} \frac{x' - y'}{||x' - y'||^{m-2j+2}} Z_k\left(\frac{(x' - y')u'(x' - y')}{||x' - y'||^2}\right) f(x', u') dS(u') dx'^m \\
&= \int_{\mathbb{R}^m} \int_{\mathbb{S}^{m-1}} \frac{y^{-1}(x-y)x^{-1}}{||y^{-1}(x-y)x^{-1}||^{m-2j+2}} \cdot Z_k\left(\frac{x(x-y)y u' y(x-y)x}{|x(x-y)y|^2}, v'\right) f(x^{-1}, u') j(x^{-1}) dS(u') dx^m \\
&= \int_{\mathbb{R}^m} \int_{\mathbb{S}^{m-1}} \left(\frac{y}{||y||^{m-2j+2}}\right)^{-1} \frac{x-y}{||x-y||^{m-2j+2}} \left(\frac{x}{||x||^{m-2j+2}}\right)^{-1} \\
&\quad \cdot -\frac{x}{||x||} Z_k\left(\frac{(x-y)u(x-y)}{||x-y||^2}, v\right) \frac{x}{||x||} f(x^{-1}, \frac{yuy}{||y||^2}) ||x||^{-2m} dS(u) dx^m \\
&= \int_{\mathbb{R}^m} \int_{\mathbb{S}^{m-1}} -\left(\frac{y}{||y||^{m-2j+2}}\right)^{-1} E_{2j-1}(x-y, u, v) \frac{x}{||x||^{m-2j}} f(x^{-1}, \frac{yuy}{||y||^2}) dS(u) dx^m \\
&= -\left(\frac{y}{||y||^{m-2j+2}}\right)^{-1} E_{2j-1}(x-y, u, v) \frac{x}{||x||^{m-2j}} * f(x^{-1}, \frac{yuy}{||y||^2}).
\end{aligned}$$

□

Hence, the intertwining operators for the convolution operators are as follows.

Proposition 8. Suppose $x \in \mathbb{R}^m$, $x' = \varphi(x) = (ax + b)(cx + d)^{-1}$ is a Möbius transformation, $u' = \frac{(cy + d)u(cy + d)}{||cy + d||^2}$, and $v' = \frac{(cx + d)v(cx + d)}{||cx + d||^2}$. Then

$$E_k(x' - y', u', v') * f(x', u') = J_k^{-1}(\varphi, y) E_k(x - y, u, v) J_{-k}(\varphi, x) * f(\varphi(x), \widetilde{\frac{(cy + d)u(cy + d)}{||cy + d||^2}}).$$

where $J_k(\varphi, x)$ and $J_{-k}(\varphi, x)$ are defined as in Proposition 4.

Recall that \mathcal{D}_k is the inverse of its fundamental solution in the sense of convolution. Hence, we obtain the intertwining operators of \mathcal{D}_k as follows.

Proposition 9. *Suppose $y \in \mathbb{R}^m$, $y' = (ay + b)(cy + d)^{-1}$ is a Möbius transformation and $u' = \frac{(cy + d)u \widetilde{(cy + d)}}{\|cy + d\|^2}$. Then*

$$\mathcal{D}_{k,y',u'} = J_{-k}^{-1}(\varphi, y) \mathcal{D}_{k,y,u} J_k(\varphi, y).$$

It is worth pointing out that for general $\alpha \in \mathbb{R}$, if we denote

$$E_k^{\alpha,1}(x - y, u, v) = \frac{x}{\|x\|^\alpha} Z_k\left(\frac{xux}{\|x\|^2}, v\right)$$

where $Z_k(u, v)$ is the reproducing kernel of \mathcal{M}_k , and

$$E_k^{\alpha,2}(x - y, u, v) = \|x\|^\alpha Z_k\left(\frac{xux}{\|x\|^2}, v\right)$$

where $Z_k(u, v)$ is the reproducing kernel of \mathcal{H}_k , then we can define a class of convolution type operators

$$\int_{\mathbb{R}^m} \int_{\mathbb{S}^{m-1}} E_k^{\alpha,i}(x - y, u, v) f_i(x, u) dS(u) dx^m$$

where $f_i(x, u) \in C^\infty(\mathbb{R}^m, U_i)$ with $U_1 = \mathcal{M}_k$ and $U_2 = \mathcal{H}_k$. More importantly, these convolution type operators are conformally invariant by similar arguments as above and their inverses, when they exist, are conformally invariant pseudo-differential operators.

4.2 Bosonic operators: even order, integer spin

With a similar strategy as in the previous section and arguing by induction, we now construct higher order conformally invariant differential operators in higher spin spaces. We start with the even order case. Denote the $2j$ -th order bosonic operator by

$$\mathcal{D}_{2j} : C^\infty(\mathbb{R}^m, \mathcal{H}_k) \longrightarrow C^\infty(\mathbb{R}^m, \mathcal{H}_k).$$

As the generalization of D_x^{2j} in Euclidean space to higher spin spaces, it is conformally invariant and has the following intertwining operators (Proposition 4):

$$\|cx + d\|^{2j+m} \mathcal{D}_{2j,y,\omega} f(y, \omega) = \mathcal{D}_{2j,x,u} \|cx + d\|^{2j-m} f(\phi(x), \frac{(cx + d)u \widetilde{(cx + d)}}{\|cx + d\|^2}),$$

where $y = \phi(x) = (ax + b)(cx + d)^{-1}$ is a Möbius transformation and $\omega = \frac{(cx + d)u \widetilde{(cx + d)}}{\|cx + d\|^2}$.

As mentioned above, the uniqueness (up to a multiplicative constant) and existence of \mathcal{D}_{2j} having the above intertwining operators can be justified by Theorems 2, 3 and 4 of [35] and Chapter 8 of [34], where the irreducible representation of $Spin(m)$ is \mathcal{H}_k .

We also have shown that (Proposition 5)

$$c_{2j}||x||^{2j-m}Z_k\left(\frac{xux}{||x||^2}, v\right),$$

is the fundamental solution of \mathcal{D}_{2j} , where c is a non-zero real constant. Therefore, to find the $2j$ -th order conformally invariant differential operator, we need only find a $2j$ -th order differential operator whose fundamental solution is $c||x||^{2j-m}Z_k\left(\frac{xux}{||x||^2}, v\right)$. Here is our first main theorem.

Theorem 3. *Let $Z_k(u, v)$ be the reproducing kernel of \mathcal{H}_k . When $j > 1$, the $2j$ -th order conformally invariant differential operator on $C^\infty(\mathbb{R}^m, \mathcal{H}_k)$ is the $2j$ -th bosonic operator*

$$\mathcal{D}_{2j} = \mathcal{D}_2 \prod_{s=2}^j \left(\mathcal{D}_2 - \frac{(2s)(2s-2)}{(m+2k-2)(m+2k-4)} \Delta_x \right)$$

that has the fundamental solution

$$a_{2j}||x||^{2j-m}Z_k\left(\frac{xux}{||x||^2}, v\right),$$

where

$$\mathcal{D}_2 = \Delta_x - \frac{4T_{k,2}T_{k,2}^*}{m+2k-2}$$

is the higher spin Laplace operator [3],

$$T_{k,2} = \langle u, D_x \rangle - \frac{||u||^2 \langle D_u, D_x \rangle}{m+2k-4} \text{ and } T_{k,2}^* = \langle D_u, D_x \rangle$$

are the second order twistor and dual twistor operators, and a_{2j} is a non-zero real constant whose expression is given later in this section.

To prove the previous theorem, we start with the following proposition.

Proposition 10. *For every $H_k(u) \in \mathcal{H}_k(\mathbb{R}^m, \mathbb{C})$, when $\alpha > 2 - m$,*

$$\left(\mathcal{D}_2 - \frac{(m+\alpha)(m+\alpha-2)}{(m+2k-2)(m+2k-4)} \Delta_x \right) ||x||^\alpha H_k\left(\frac{xux}{||x||^2}\right) = c_{\alpha+m} ||x||^{\alpha-2} H_k\left(\frac{xux}{||x||^2}\right),$$

in the distribution sense, where

$$c_{\alpha+m} = -(m+\alpha)(m+\alpha-2) \frac{(\alpha-2k)(\alpha-2k-2) + 2k(m+2\alpha-2k-4)}{(m+2k-2)(m+2k-4)}.$$

Proof. In order to prove the above proposition with an arbitrary function $H_k(u) \in \mathcal{H}_k$, as stated in [3], we can rely on the fact \mathcal{H}_k is an irreducible $Spin(m)$ -representation

generated by the highest weight vector $\langle u, 2\mathbf{f}_1 \rangle^k$. As \mathcal{D}_2 and Δ_x are both $Spin(m)$ -invariant operators, it suffices to prove the statement for

$$||x||^\alpha \langle \frac{xux}{||x||^2}, 2\mathbf{f}_1 \rangle^k = ||x||^{\alpha-2k} \langle xux, 2\mathbf{f}_1 \rangle^k = ||x||^{\alpha-2k} \langle u||x||^2 - 2\langle u, x \rangle x, 2\mathbf{f}_1 \rangle^k.$$

First, we assume $x \neq 0$. On the one hand, we have

$$\begin{aligned} \Delta_x ||x||^{\alpha-2k} \langle xux, 2\mathbf{f}_1 \rangle^k &= \Delta_x ||x||^{\alpha-2k} \langle u||x||^2 - 2\langle u, x \rangle x, 2\mathbf{f}_1 \rangle^k \\ &= \Delta_x (||x||^{\alpha-2k}) \langle xux, 2\mathbf{f}_1 \rangle^k + ||x||^{\alpha-2k} \Delta_x (\langle xux, 2\mathbf{f}_1 \rangle^k) + \sum_{j=1}^m \partial_{x_j} (||x||^{\alpha-2k}) \partial_{x_j} (\langle xux, 2\mathbf{f}_1 \rangle^k). \end{aligned}$$

Since

$$\partial_{x_j} \langle xux, 2\mathbf{f}_1 \rangle^k = \partial_{x_j} \langle u||x||^2 - 2\langle u, x \rangle x, 2\mathbf{f}_1 \rangle^k = k \langle xux, 2\mathbf{f}_1 \rangle^{k-1} \langle 2ux_j - 2u_j x - 2\langle u, x \rangle e_j, 2\mathbf{f}_1 \rangle,$$

and from [3]

$$\Delta_x \langle xux, 2\mathbf{f}_1 \rangle^k = 4k(k-1) ||u||^2 \langle x, 2\mathbf{f}_1 \rangle^2 \langle xux, 2\mathbf{f}_1 \rangle^{k-2} + 2k(m+2k-4) \langle u, 2\mathbf{f}_1 \rangle \langle xux, 2\mathbf{f}_1 \rangle^{k-1}.$$

Therefore,

$$\begin{aligned} &\Delta_x ||x||^{\alpha-2k} \langle xux, 2\mathbf{f}_1 \rangle^k \\ &= [(\alpha-2k)(\alpha-2k-2) + 2k(m-2k+2\alpha-4)] ||x||^{\alpha-2k-2} \langle xux, 2\mathbf{f}_1 \rangle^k \\ &\quad + 4k(m+2k-4) \langle u, x \rangle \langle x, 2\mathbf{f}_1 \rangle ||x||^{\alpha-2k-2} \langle xux, 2\mathbf{f}_1 \rangle^{k-1} \\ &\quad + 4k(k-1) ||u||^2 \langle x, 2\mathbf{f}_1 \rangle^2 ||x||^{\alpha-2k} \langle xux, 2\mathbf{f}_1 \rangle^{k-2}. \end{aligned}$$

On the other hand, we have [3]

$$\begin{aligned} \mathcal{D}_2 ||x||^{\alpha-2k} \langle xux, 2\mathbf{f}_1 \rangle^k &= (m+\alpha-2) \left(\alpha + \frac{4k}{m+2k-2} \right) ||x||^{\alpha-2k-2} \langle xux, 2\mathbf{f}_1 \rangle^k \\ &\quad + (m+\alpha-2)(m+\alpha) \frac{4k}{m+2k-2} \langle u, x \rangle \langle x, 2\mathbf{f}_1 \rangle ||x||^{\alpha-2k-2} \langle xux, 2\mathbf{f}_1 \rangle^{k-1} \\ &\quad + \frac{4k(k-1)(m+\alpha)(m+\alpha-2)}{(m+2k-2)(m+2k-4)} ||u||^2 \langle x, 2\mathbf{f}_1 \rangle^2 ||x||^{\alpha-2k} \langle xux, 2\mathbf{f}_1 \rangle^{k-2}. \end{aligned}$$

Combining the above two equalities completes the proof when $x \neq 0$. Next, we consider the singularity of $\phi(x, u)$ at $x = 0$. Notice that singularity only occurs in the $||x||^\alpha$ part and that $||x||^\alpha$ is weak differentiable if $\alpha > -m+1$ with weak derivative $\partial_{x_i} ||x||^\alpha = \alpha x_i ||x||^{\alpha-2}$. Hence, with the assumption that $\alpha > 2-m$, every differentiation in the process above is also correct in the distribution sense. This completes the proof. \square

Now, we can prove the following proposition immediately.

Proposition 11. When integer $j > 1$,

$$\mathcal{D}_{2j}a_{2j}||x||^{2j-m}H_k\left(\frac{xux}{||x||^2}\right) = \delta(x)H_k(u),$$

where $H_k(u) \in \mathcal{H}_k(\mathbb{R}^m, \mathbb{C})$ and

$$a_{2j} = \frac{(m+2k-4)\Gamma(\frac{m}{2}-1)}{4(4-m)\pi^{\frac{m}{2}}} \prod_{s=2}^j c_{2s}^{-1},$$

c_{2s} defined by Proposition 10 for $\alpha = 2s - m$.

Proof. We prove this proposition by induction. First, when $j = 2$,

$$\begin{aligned} & \mathcal{D}_4a_4||x||^{4-m}H_k\left(\frac{xux}{||x||^2}\right) \\ &= (\mathcal{D}_2 - \frac{8}{(m+2k-2)(m+2k-4)}\Delta_x)\mathcal{D}_2a_4||x||^{4-m}H_k\left(\frac{xux}{||x||^2}\right) \\ &= \mathcal{D}_2(\mathcal{D}_2 - \frac{8}{(m+2k-2)(m+2k-4)}\Delta_x)a_4||x||^{4-m}H_k\left(\frac{xux}{||x||^2}\right) \\ &= \mathcal{D}_2\frac{(m+2k-4)\Gamma(\frac{m}{2}-1)}{4(4-m)\pi^{\frac{m}{2}}}||x||^{2-m}H_k\left(\frac{xux}{||x||^2}\right), \end{aligned}$$

where the last line follows using $\alpha = 4 - m$ in Proposition 10. Thanks to Theorem 5.1 in [3], this last equation is equal to $\delta(x)H_k(u)$.

Assume when $j = s$ that the proposition is true. Then for $j = s + 1$, we have

$$\begin{aligned} & \mathcal{D}_{2s+2}a_{2s+2}||x||^{2s+2-m}H_k\left(\frac{xux}{||x||^2}\right) \\ &= (\mathcal{D}_2 - \frac{2s(2s+2)}{(m+2k-2)(m+2k-4)}\Delta_x)\mathcal{D}_{2s}a_{2s}c_{2s+2}^{-1}||x||^{6-m}H_k\left(\frac{xux}{||x||^2}\right) \\ &= \mathcal{D}_{2s}(\mathcal{D}_2 - \frac{24}{(m+2k-2)(m+2k-4)}\Delta_x)c_{2s+2}^{-1}a_{2s}||x||^{2s+2-m}H_k\left(\frac{xux}{||x||^2}\right) \\ &= \mathcal{D}_{2s}a_{2s}||x||^{2s-m}H_k\left(\frac{xux}{||x||^2}\right) = \delta(x)H_k(u), \end{aligned}$$

where the penultimate equality follows using $\alpha = 2s + 2 - m$ in Proposition 10. This last equation comes from our assumption $j = s$. Therefore, our proposition is proved. \square

In particular, from the above proposition, we have

$$\mathcal{D}_{2j}a_{2j}||x||^{2j-m}Z_k\left(\frac{xux}{||x||^2}, v\right) = \delta(x)Z_k(u, v),$$

where $Z_k(u, v)$ is the reproducing kernel of \mathcal{H}_k . Hence, Theorem 3 is proved and the even order case is resolved.

4.3 Fermionic operators: odd order, half-integer spin

We denote the $(2j - 1)$ -th fermionic operator

$$\mathcal{D}_{2j-1} : C^\infty(\mathbb{R}^m, \mathcal{M}_k) \longrightarrow C^\infty(\mathbb{R}^m, \mathcal{M}_k).$$

as the generalization of D_x^{2j-1} in Euclidean space to higher spin spaces. With similar arguments as in bosonic case, it is conformally invariant with the following intertwining operators

$$\frac{\widetilde{cx+d}}{\|cx+d\|^{m+2j}} \mathcal{D}_{2j-1, y, \omega} f(y, \omega) = \mathcal{D}_{2j-1, x, u} \frac{\widetilde{cx+d}}{\|cx+d\|^{m-2j+2}} f(\phi(x), \frac{(cx+d)u\widetilde{cx+d}}{\|cx+d\|^2}),$$

where $y = \phi(x) = (ax+b)(cx+d)^{-1}$ is a Möbius transformation and $\omega = \frac{(cx+d)u\widetilde{cx+d}}{\|cx+d\|^2}$.

Furthermore, its fundamental solution is

$$c_{2j-1} \frac{x}{\|x\|^{m-2j+2}} Z_k(\frac{xux}{\|x\|^2}, v),$$

where c_{2j-1} is a non-zero real constant and $Z_k(u, v)$ is the reproducing kernel of \mathcal{M}_k . Here comes our second main theorem. The proof is left in Section 5.

Theorem 4. *Let $Z_k(u, v)$ be the reproducing kernel of \mathcal{M}_k . When $j > 1$, the $(2j - 1)$ -th order conformally invariant differential operator on $C^\infty(\mathbb{R}^m, \mathcal{M}_k)$ is the $(2j - 1)$ -th order fermionic operator*

$$\mathcal{D}_{2j-1} = R_k \prod_{s=1}^{j-1} \left(-R_k^2 + \frac{4s^2 T_k^* T_k}{(m+2k-2s-2)(m+2k+2s-2)} \right)$$

that has the fundamental solution

$$\lambda_{2s} \frac{x}{\|x\|^{m-2j+2}} Z_k(\frac{xux}{\|x\|^2}, v),$$

where

$$T_k = (1 + \frac{uD_u}{m+2k-2})D_x \text{ and } T_k^* = \frac{-uD_u D_x}{m+2k-2}$$

are the twistor and dual twistor operators defined in [9] and λ_{2s} is a non-zero real constant whose expression is given in Section 5.

On a concluding note, recall that a manifold is conformally flat if it has an atlas whose transition functions are Möbius transformations. Note this does not involve curvature. Using similar arguments as in our previous paper [12], one can generalize our conformally invariant differential operators to conformally flat spin manifolds in the fermionic case and conformally flat Riemannian manifolds in the bosonic case. More specifically, from Proposition 4, if we choose a particular Möbius transformation, we can generalize our fermionic and bosonic operators to some conformally flat manifold, such as the unit sphere, using Equation (2). This will be developed more formally elsewhere.

5 Explicit proof for the construction of \mathcal{D}_{2j-1}

To prove Theorem 4, we start with the following proposition.

Proposition 12. *For any $f_k(u) \in \mathcal{M}_k$, we denote*

$$B_{m-\beta} = \Delta_x + a_{m-\beta} \|u\|^2 \langle D_u, D_x \rangle^2 + b_{m-\beta} \langle u, D_x \rangle \langle D_u, D_x \rangle + c_{m-\beta} u \langle D_u, D_x \rangle D_x.$$

When $\beta \leq m - 2$, we have

$$B_{m-\beta} \frac{x}{\|x\|^\beta} f_k\left(\frac{xux}{\|x\|^2}\right) = d_{m-\beta} \frac{x}{\|x\|^{\beta+2}} f_k\left(\frac{xux}{\|x\|^2}\right)$$

in the distribution sense, where

$$\begin{aligned} a_{m-\beta} &= \frac{4}{(\beta + 2k - 2)(2m + 2k - \beta - 2)}; \\ b_{m-\beta} &= -\frac{4(m + 2k - 2)}{(\beta + 2k - 2)(2m + 2k - \beta - 2)}; \\ c_{m-\beta} &= -\frac{4}{(\beta + 2k - 2)(2m + 2k - \beta - 2)}; \\ d_{m-\beta} &= (\beta + 2k)(\beta + 2k - m) + 2k(m - 2\beta - 2k - 2) + \frac{4k(m + 2k - 2)}{\beta + 2k - 2}. \end{aligned}$$

It is worth pointing out that if $\beta = m - 2s$, then B_{2s} is exactly the term in the parenthesis in Theorem 4. Details can be found later in this section.

In order to prove the above proposition with arbitrary function $f_k(u) \in \mathcal{M}_k$, as stated in [3], we can rely on the fact \mathcal{M}_k is an irreducible $Spin(m)$ -representation generated by the highest weight vector $\langle u, 2\mathfrak{f}_1 \rangle^k I$, where I is defined in Section 2.2.1. It suffices to prove the statement for

$$\frac{x}{\|x\|^\beta} \langle \frac{xux}{\|x\|^2}, 2\mathfrak{f}_1 \rangle^k I = \frac{x}{\|x\|^{\beta+2k}} \langle xux, 2\mathfrak{f}_1 \rangle^k I = \frac{x}{\|x\|^{\beta+2k}} \langle u\|x\|^2 - 2\langle u, x \rangle x, 2\mathfrak{f}_1 \rangle^k I.$$

First, we assume that $x \neq 0$, and we have the following technical lemmas.

Lemma 1.

$$\begin{aligned} & \Delta_x \frac{x}{\|x\|^{\beta+2k}} \langle xux, 2\mathfrak{f}_1 \rangle^k I \\ &= [(\beta + 2k)(\beta + 2k - m) + 2k(m - 2\beta - 2k - 2)] \frac{x}{\|x\|^{\beta+2k+2}} \langle xux, 2\mathfrak{f}_1 \rangle^k I \\ & \quad - 4k \frac{u \langle x, 2\mathfrak{f}_1 \rangle}{\|x\|^{\beta+2k}} \langle xux, 2\mathfrak{f}_1 \rangle^{k-1} I + 4k(m + 2k - 2) \frac{x}{\|x\|^{\beta+2k+2}} \langle u, x \rangle \langle x, 2\mathfrak{f}_1 \rangle \langle xux, 2\mathfrak{f}_1 \rangle^{k-1} I \\ & \quad + 4k(k - 1) \|u\|^2 \langle x, 2\mathfrak{f}_1 \rangle^2 \frac{x}{\|x\|^{\beta+2k}} \langle xux, 2\mathfrak{f}_1 \rangle^{k-2} I. \end{aligned}$$

Proof. Since

$$\Delta_x \frac{x}{||x||^{\beta+2k}} = (\beta + 2k)(\beta + 2k - m) \frac{x}{||x||^{\beta+2k+2}}$$

and [3] gives

$$\Delta_x \langle xux, 2\mathbf{f}_1 \rangle^k = 4k(k-1)||u||^2 \langle x, 2\mathbf{f}_1 \rangle^2 \langle xux, 2\mathbf{f}_1 \rangle^{k-2} I + 2k(m+2k-4) \langle u, 2\mathbf{f}_1 \rangle \langle xux, 2\mathbf{f}_1 \rangle^{k-1} I,$$

we have

$$\begin{aligned} & \Delta_x \frac{x}{||x||^{\beta+2k}} \langle xux, 2\mathbf{f}_1 \rangle^k I \\ = & \Delta_x \left(\frac{x}{||x||^{\beta+2k}} \right) \langle xux, 2\mathbf{f}_1 \rangle^k I + \frac{x}{||x||^{\beta+2k}} \Delta_x (\langle xux, 2\mathbf{f}_1 \rangle^k) I + 2 \sum_{i=1}^m \partial_{x_i} \frac{x}{||x||^{\beta+2k}} \partial_{x_i} \langle xux, 2\mathbf{f}_1 \rangle^k I \\ = & (\beta + 2k)(\beta + 2k - m) \frac{x}{||x||^{\beta+2k+2}} \langle xux, 2\mathbf{f}_1 \rangle^k I \\ & + \frac{x}{||x||^{\beta+2k}} (4k(k-1)||u||^2 \langle x, 2\mathbf{f}_1 \rangle^2 \langle xux, 2\mathbf{f}_1 \rangle^{k-2} I + 2k(m+2k-4) \langle u, 2\mathbf{f}_1 \rangle \langle xux, 2\mathbf{f}_1 \rangle^{k-1} I) \\ & + 2k \sum_{i=1}^m \left(\frac{e_i}{||x||^{\beta+2k}} - \frac{(\beta + 2k)x_i x}{||x||^{\beta+2k+2}} \right) \langle 2ux_i - 2u_i x - 2\langle u, x \rangle e_i, 2\mathbf{f}_1 \rangle \langle xux, 2\mathbf{f}_1 \rangle^{k-1} I. \end{aligned}$$

Notice that $I = \mathbf{f}_1 \mathbf{f}_1^\dagger \mathbf{f}_2 \mathbf{f}_2^\dagger \cdots \mathbf{f}_n \mathbf{f}_n^\dagger$ and $\mathbf{f}_1^2 = 0$. Therefore, we obtain

$$\begin{aligned} & = (\beta + 2k)(\beta + 2k - m) \frac{x}{||x||^{\beta+2k+2}} \langle xux, 2\mathbf{f}_1 \rangle^k I \\ & + 4k(k-1)||u||^2 \langle x, 2\mathbf{f}_1 \rangle^2 \frac{x}{||x||^{\beta+2k}} \langle xux, 2\mathbf{f}_1 \rangle^{k-2} I \\ & + 2k(m-2\beta-2k-2) \frac{x}{||x||^{\beta+2k}} \langle u, 2\mathbf{f}_1 \rangle \langle xux, 2\mathbf{f}_1 \rangle^{k-1} I \\ & - 4k \frac{u \langle x, 2\mathbf{f}_1 \rangle}{||x||^{\beta+2k}} \langle xux, 2\mathbf{f}_1 \rangle^{k-1} I + 8k(\beta + 2k) \frac{x \langle u, x \rangle \langle x, 2\mathbf{f}_1 \rangle}{||x||^{\beta+2k+2}} \langle xux, 2\mathbf{f}_1 \rangle^{k-1} I. \end{aligned}$$

With the help of $\langle u||x||^2, 2\mathbf{f}_1 \rangle = \langle xux, 2\mathbf{f}_1 \rangle + 2\langle u, x \rangle \langle x, 2\mathbf{f}_1 \rangle$, this lemma is proved immediately. \square

Lemma 2.

$$\begin{aligned} & ||u||^2 \langle D_u, D_x \rangle^2 \frac{x}{||x||^{\beta+2k}} \langle xux, 2\mathbf{f}_1 \rangle^k I \\ = & k(k-1)(2m-\beta+2k-2)(2m-\beta+2k-4) \frac{||u||^2 x}{||x||^{\beta+2k}} \langle x, 2\mathbf{f}_1 \rangle^2 \langle xux, 2\mathbf{f}_1 \rangle^{k-2} I; \quad (4) \end{aligned}$$

$$\begin{aligned} & u \langle D_u, D_x \rangle D_x \frac{x}{||x||^{\beta+2k}} \langle xux, 2\mathbf{f}_1 \rangle^k I \\ = & -k(2m-\beta+2k-2) \left[(\beta-m) \frac{u \langle x, 2\mathbf{f}_1 \rangle}{||x||^{\beta+2k}} \langle xux, 2\mathbf{f}_1 \rangle^{k-1} I \right. \\ & \left. + 2(k-1) ||u||^2 \frac{x}{||x||^{\beta+2k}} \langle x, 2\mathbf{f}_1 \rangle^2 \langle xux, 2\mathbf{f}_1 \rangle^{k-2} I \right]; \quad (5) \end{aligned}$$

$$\begin{aligned} & \langle u, D_x \rangle \langle D_u, D_x \rangle \frac{x}{||x||^{\beta+2k}} \langle xux, 2\mathbf{f}_1 \rangle^k I \\ = & -k(2m-\beta+2k-2) \left[\frac{x \langle xux, 2\mathbf{f}_1 \rangle^k}{||x||^{\beta+2k+2}} I - (\beta+2k-2) \frac{x \langle u, x \rangle \langle x, 2\mathbf{f}_1 \rangle}{||x||^{\beta+2k+2}} \langle xux, 2\mathbf{f}_1 \rangle^{k-1} I \right. \\ & \left. + \frac{u \langle x, 2\mathbf{f}_1 \rangle}{||x||^{\beta+2k}} \langle xux, 2\mathbf{f}_1 \rangle^{k-1} I - 2(k-1) \frac{||u||^2 x}{||x||^{\beta+2k}} \langle x, 2\mathbf{f}_1 \rangle^2 \langle xux, 2\mathbf{f}_1 \rangle^{k-2} I \right] \quad (6) \end{aligned}$$

Proof. Since these three operators on the left contain $\langle D_u, D_x \rangle$, first let us check:

$$\begin{aligned} & \langle D_u, D_x \rangle \frac{x}{||x||^{\beta+2k}} \langle xux, 2\mathbf{f}_1 \rangle^k I = \sum_{i=1}^m \partial_{u_i} \left(\left(\frac{e_i}{||x||^{\beta+2k}} - \frac{(\beta+2k)x_i x}{||x||^{\beta+2k+2}} \right) \langle xux, 2\mathbf{f}_1 \rangle^k I \right. \\ & \left. + k \frac{x}{||x||^{\beta+2k}} \langle xux, 2\mathbf{f}_1 \rangle^{k-1} \langle 2ux_i - 2u_i x - 2\langle u, x \rangle e_i, 2\mathbf{f}_1 \rangle I \right) \\ = & \sum_{i=1}^m k \left(\frac{e_i}{||x||^{\beta+2k}} - \frac{(\beta+2k)x_i x}{||x||^{\beta+2k+2}} \right) \langle xux, 2\mathbf{f}_1 \rangle^{k-1} \langle e_i ||x||^2 - 2x_i x, 2\mathbf{f}_1 \rangle I \\ & + \sum_{i=1}^m k(k-1) \frac{x \langle xux, 2\mathbf{f}_1 \rangle^{k-2}}{||x||^{\beta+2k}} \langle e_i ||x||^2 - 2x_i x, 2\mathbf{f}_1 \rangle \langle 2ux_i - 2u_i x - 2\langle u, x \rangle e_i, 2\mathbf{f}_1 \rangle I \\ & + \sum_{i=1}^m k \frac{x \langle xux, 2\mathbf{f}_1 \rangle^{k-1}}{||x||^{\beta+2k}} \langle 2e_i x_i - 2x - 2x_i e_i, 2\mathbf{f}_1 \rangle I. \end{aligned}$$

The last expression simplifies as

$$-k(2m-\beta+2k-2) \frac{x \langle x, 2\mathbf{f}_1 \rangle}{||x||^{\beta+2k}} \langle xux, 2\mathbf{f}_1 \rangle^{k-1} I.$$

Hence, to verify Eq. (4), we only need to check

$$\begin{aligned}
& \langle D_u, D_x \rangle \frac{x \langle x, 2\mathbf{f}_1 \rangle}{\|x\|^{\beta+2k}} \langle xux, 2\mathbf{f}_1 \rangle^{k-1} I = \sum_{i=1}^m \partial_{u_i} \left(\left(\frac{e_i}{\|x\|^{\beta+2k}} - \frac{(\beta+2k)x_i x}{\|x\|^{\beta+2k+2}} \right) \langle x, 2\mathbf{f}_1 \rangle \langle xux, 2\mathbf{f}_1 \rangle^{k-1} I \right. \\
& \quad \left. + \frac{x \langle e_i, 2\mathbf{f}_1 \rangle}{\|x\|^{\beta+2k}} \langle xux, 2\mathbf{f}_1 \rangle^{k-1} I + (k-1) \frac{x \langle x, 2\mathbf{f}_1 \rangle}{\|x\|^{\beta+2k}} \langle xux, 2\mathbf{f}_1 \rangle^{k-2} \langle 2ux_i - 2u_i x - 2\langle u, x \rangle e_i, 2\mathbf{f}_1 \rangle I \right) \\
& = \sum_{i=1}^m \left(\left(\frac{e_i}{\|x\|^{\beta+2k}} - \frac{(\beta+2k)x_i x}{\|x\|^{\beta+2k+2}} \right) \langle x, 2\mathbf{f}_1 \rangle (k-1) \langle xux, 2\mathbf{f}_1 \rangle^{k-2} \langle e_i \|x\|^2 - 2x_i x, 2\mathbf{f}_1 \rangle I \right. \\
& \quad + \sum_{i=1}^m \frac{x \langle e_i, 2\mathbf{f}_1 \rangle}{\|x\|^{\beta+2k}} (k-1) \langle xux, 2\mathbf{f}_1 \rangle^{k-2} \langle e_i \|x\|^2 - 2x_i x, 2\mathbf{f}_1 \rangle I \\
& \quad + \sum_{i=1}^m (k-1) \frac{x \langle x, 2\mathbf{f}_1 \rangle}{\|x\|^{\beta+2k}} \langle xux, 2\mathbf{f}_1 \rangle^{k-2} \langle 2e_i x_i - 2x - 2x_i e_i, 2\mathbf{f}_1 \rangle I \\
& \quad \left. + \sum_{i=1}^m (k-1) \frac{x \langle x, 2\mathbf{f}_1 \rangle}{\|x\|^{\beta+2k}} (k-2) \langle xux, 2\mathbf{f}_1 \rangle^{k-3} \langle 2ux_i - 2u_i x - 2\langle u, x \rangle e_i, 2\mathbf{f}_1 \rangle \langle e_i \|x\|^2 - 2x_i x, 2\mathbf{f}_1 \rangle I \right).
\end{aligned}$$

This last expression simplifies as

$$-(k-1)(2m-\beta+2k-4) \frac{x \langle x, 2\mathbf{f}_1 \rangle^2}{\|x\|^{\beta+2k}} \langle xux, 2\mathbf{f}_1 \rangle^{k-2} I.$$

Hence, Eq. (4) is verified.

For Eq. (5), we check

$$\begin{aligned}
& u D_x \frac{x \langle x, 2\mathbf{f}_1 \rangle}{\|x\|^{\beta+2k}} \langle xux, 2\mathbf{f}_1 \rangle^{k-1} I = u \sum_{i=1}^m e_i \left(\left(\frac{e_i}{\|x\|^{\beta+2k}} - \frac{(\beta+2k)x_i x}{\|x\|^{\beta+2k+2}} \right) \langle x, 2\mathbf{f}_1 \rangle \langle xux, 2\mathbf{f}_1 \rangle^{k-1} I \right. \\
& \quad \left. + \frac{x \langle e_i, 2\mathbf{f}_1 \rangle}{\|x\|^{\beta+2k}} \langle xux, 2\mathbf{f}_1 \rangle^{k-1} I + (k-1) \frac{x \langle x, 2\mathbf{f}_1 \rangle}{\|x\|^{\beta+2k}} \langle xux, 2\mathbf{f}_1 \rangle^{k-2} \langle 2ux_i - 2u_i x - 2\langle u, x \rangle e_i, 2\mathbf{f}_1 \rangle I \right) \\
& = u \left[\frac{(\beta+2k-m) \langle x, 2\mathbf{f}_1 \rangle}{\|x\|^{\beta+2k}} \langle xux, 2\mathbf{f}_1 \rangle^{k-1} I - 2 \frac{\langle x, 2\mathbf{f}_1 \rangle}{\|x\|^{\beta+2k}} \langle xux, 2\mathbf{f}_1 \rangle^{k-1} I \right. \\
& \quad - 2(k-1) \frac{\langle x, 2\mathbf{f}_1 \rangle \langle u \|x\|^2, 2\mathbf{f}_1 \rangle}{\|x\|^{\beta+2k}} \langle xux, 2\mathbf{f}_1 \rangle^{k-2} I \\
& \quad \left. - 2(k-1) \frac{ux}{\|x\|^{\beta+2k}} \langle x, 2\mathbf{f}_1 \rangle^2 \langle xux, 2\mathbf{f}_1 \rangle^{k-2} I + 4(k-1) \frac{\langle u, x \rangle \langle x, 2\mathbf{f}_1 \rangle^2}{\|x\|^{\beta+2k}} \langle xux, 2\mathbf{f}_1 \rangle^{k-2} I \right] \\
& = (\beta-m) \frac{u \langle x, 2\mathbf{f}_1 \rangle}{\|x\|^{\beta+2k}} \langle xux, 2\mathbf{f}_1 \rangle^{k-1} I + 2(k-1) \frac{\|u\|^2 x}{\|x\|^{\beta+2k}} \langle x, 2\mathbf{f}_1 \rangle^2 \langle xux, 2\mathbf{f}_1 \rangle^{k-2} I.
\end{aligned}$$

For Eq. (6), we check

$$\begin{aligned}
& \langle u, D_x \rangle \frac{x \langle x, 2\mathbf{f}_1 \rangle}{\|x\|^{\beta+2k}} \langle xux, 2\mathbf{f}_1 \rangle^{k-1} I = \sum_{i=1}^m u_i \left(\left(\frac{e_i}{\|x\|^{\beta+2k}} - \frac{(\beta+2k)x_i x}{\|x\|^{\beta+2k+2}} \right) \langle x, 2\mathbf{f}_1 \rangle \langle xux, 2\mathbf{f}_1 \rangle^{k-1} I \right. \\
& \quad \left. + \frac{x \langle e_i, 2\mathbf{f}_1 \rangle}{\|x\|^{\beta+2k}} \langle xux, 2\mathbf{f}_1 \rangle^{k-1} + (k-1) \frac{x \langle x, 2\mathbf{f}_1 \rangle}{\|x\|^{\beta+2k}} \langle xux, 2\mathbf{f}_1 \rangle^{k-2} \langle 2ux_i - 2u_i x - 2\langle u, x \rangle e_i, 2\mathbf{f}_1 \rangle I \right) \\
& = \frac{x \langle xux, 2\mathbf{f}_1 \rangle^k}{\|x\|^{\beta+2k+2}} I - (\beta+2k-2) \frac{x \langle u, x \rangle \langle x, 2\mathbf{f}_1 \rangle}{\|x\|^{\beta+2k-2}} \langle xux, 2\mathbf{f}_1 \rangle^{k-1} I \\
& \quad + \frac{u \langle x, 2\mathbf{f}_1 \rangle}{\|x\|^{\beta+2k}} \langle xux, 2\mathbf{f}_1 \rangle^{k-1} I - 2(k-1) \frac{\|u\|^2 x}{\|x\|^{\beta+2k}} \langle x, 2\mathbf{f}_1 \rangle^2 \langle xux, 2\mathbf{f}_1 \rangle^{k-2} I.
\end{aligned}$$

Therefore, Eqs. (5) and (6) are verified. \square

Recall the fact mentioned in the $2j$ -th order bosonic operator case, $\|x\|^\alpha$ is weak differentiable if $\alpha > -m+1$ with weak derivative $\partial_{x_i} \|x\|^\alpha = \alpha x_i \|x\|^{\alpha-2}$. Hence, when $\beta \leq m-2$, Lemmas 1 and 2 are both true in the distribution sense. Combining them completes the proof of Proposition 12. With the help of Proposition 12 and similar arguments as in Proposition 11, we have the following proposition by induction.

Proposition 13. *Let $f_k(u) \in \mathcal{M}_k$. When integer $j > 1$,*

$$\left[\prod_{s=1}^{j-1} B_{2s} d_{2s}^{-1} \right] \frac{x}{\|x\|^{m-2j+2}} f_k \left(\frac{xux}{\|x\|^2} \right) = \frac{x}{\|x\|^m} f_k \left(\frac{xux}{\|x\|^2} \right)$$

in the distribution sense, where a_{2s} , b_{2s} , c_{2s} , d_{2s} are defined as in Proposition 12 with $\beta = m-2s$.

In the above proposition, it is worth pointing out that

$$B_{2s_1} B_{2s_2} = B_{2s_2} B_{2s_1},$$

where $s_1 \neq s_2$. Indeed, with a straightforward calculation, one can get

$$R_k^2 = -\Delta_x + \frac{4\langle u, D_x \rangle \langle D_u, D_x \rangle}{m+2k-2} - \frac{4\|u\|^2 \langle D_u, D_x \rangle^2}{(m+2k-2)^2} + \frac{4u \langle D_u, D_x \rangle D_x}{(m+2k-2)^2}.$$

Then

$$\begin{aligned}
B_{2s} &= \Delta_x + \frac{4(\|u\|^2 \langle D_u, D_x \rangle^2 - (m+2k-2) \langle u, D_x \rangle \langle D_u, D_x \rangle^2 - u \langle D_u, D_x \rangle D_x)}{(m+2k-2s-2)(m+2k+2s-2)} \\
&= \Delta_x - \frac{(m+2k-2)^2}{(m+2k-2s-2)(m+2k+2s-2)} (R_k^2 + \Delta_x).
\end{aligned} \tag{7}$$

So B_{2s} is a linear combination of R_k^2 and Δ_x . This is no surprise, since [15] points out

$$\{R_k^i \Delta_x^j, 0 \leq i \leq \min(2p+1, 2k+1), 0 \leq j, i+2j=p\}$$

is the basis of the space of $Spin(m)$ -invariant constant coefficient differential operators of order p on \mathcal{M}_k . \mathcal{D}_{2j-1} is conformally invariant, so it is also $Spin(m)$ -invariant and hence can be expressed in this basis. Furthermore, with the help of $-\Delta_x = R_k^2 + T_k T_k^*$ and Eq. (7), we can also rewrite B_{2s} in terms of first order conformally invariant operators:

$$B_{2s} = -R_k^2 + \frac{4s^2 T_k T_k^*}{(m+2k-2s-2)(m+2k+2s-2)}.$$

Now, we have fundamental solution of \mathcal{D}_{2j-1} restated as follows.

Theorem. *Let $Z_k(u, v)$ be the reproducing kernel of \mathcal{M}_k . When $j > 1$, the $(2j-1)$ -th order fermionic operator \mathcal{D}_{2j-1} has fundamental solution*

$$\lambda_{2s} \frac{x}{\|x\|^{m-2j+2}} Z_k\left(\frac{xux}{\|x\|^2}, v\right), \quad \lambda_{2s} = \frac{-(m+2k-2)}{(m-2)\omega_{m-1}} \prod_{s=1}^{j-1} d_{2s}^{-1},$$

where d_{2s} is defined in Proposition 12 with $\beta = m-2s$ and ω_{m-1} is the area of $(m-1)$ -dimensional unit sphere.

Proof. With the help of Proposition 13 and noticing that

$$\frac{-(m+2k-2)}{(m-2)\omega_{m-1}} \frac{x}{\|x\|^m} Z_k\left(\frac{xux}{\|x\|^2}, v\right)$$

is the fundamental solution of R_k [6], the above theorem follows immediately. \square

Hence, Theorem 4 is proved and the odd order case is resolved.

References

- [1] L.V. Ahlfors, *Möbius transformations in \mathbb{R}^n expressed through 2×2 matrices of Clifford numbers*, Complex Variables, Vol. 5, 1986, pp. 215-224.
- [2] M.F. Atiyah, R. Bott, A. Shapiro, *Clifford modules*, Topology, Vol. 3, Suppl. 1, 1964, pp.3-38.
- [3] H. De Bie, David Eelbode, Matthias Roels, *The higher spin Laplace operator*, arXiv:1501.03974 [math-ph]
- [4] F. Brackx, R. Delanghe, F. Sommen, *Clifford Analysis*, Pitman, London, 1982.
- [5] F. Brackx, D. Eelbode, L. Van de Voorde, *Higher spin Dirac operators between spaces of simplicial monogenics in two vector variables*, Mathematical Physics, Analysis and Geometry, Vol. 14, Issue 1, 2011, pp. 1-20.

- [6] J. Bureš, F. Sommen, V. Souček, P. Van Lancker, *Rarita-Schwinger Type Operators in Clifford Analysis*, J. Funct. Anal. Vol. 185, No. 2, 2001, pp. 425-455.
- [7] J.L. Clerc, B. Orsted, *Conformal covariance for the powers of the Dirac operator*, <https://arxiv.org/abs/1409.4983>
- [8] R. Delanghe, F. Sommen, V. Souček, *Clifford Algebra and Spinor-Valued Functions: A Function Theory for the Dirac Operator*, Kluwer, Dordrecht, 1992.
- [9] C. Ding, J. Ryan, *On Some Conformally Invariant Operators in Euclidean Space*, arXiv:1509.00098 [math.CV], submitted.
- [10] C. Ding, R. Walter, *Third order fermionic and fourth order bosonic operators*, arXiv:1602.04202 [math.DG], submitted.
- [11] C. Ding, R. Walter, J. Ryan, *Higher order fermionic and bosonic operators*, arXiv:1512.07322 [math.DG], submitted.
- [12] C. Ding, R. Walter, J. Ryan, *Higher order fermionic and bosonic operators on cylinders and Hopf manifolds*, Journal of Indian Mathematical Society, Vol. 83, Issue 3-4, 2016, pp. 231-240.
- [13] C. F. Dunkl, J. Li, J. Ryan, P. Van Lancker, *Some Rarita-Schwinger type operators*, Computational Methods and Function Theory, Vol. 13, Issue 3, 2013, pp. 397-424.
- [14] M. Eastwood, *The Cartan product*, Bulletin of the Belgian Mathematical Society, Vol. 11, Issue 5, 2005, pp. 641-651.
- [15] D. Eelbode, D. Šmíd, *Algebra of Invariants for the Rarita-Schwinger Operators*, Ann. Ac. Sci. Fennicae, Vol. 34, 2009, pp. 637-649.
- [16] D. Eelbode, M. Roels, *Generalised Maxwell equations in higher dimensions*, Complex Analysis and Operator Theory, Vol. 10, Issue 2, 2016, pp. 267-293.
- [17] H.D. Fegan, *Conformally invariant first order differential operators*. Quart. J. Math. 27, 1976, pp. 371-378.
- [18] William Fulton and Joe Harris, *Representation theory. A first course*, Graduate Texts in Mathematics, Readings in Mathematics 129, New York: Springer-Verlag, 1991.
- [19] J. Gilbert and M. Murray, *Clifford Algebras and Dirac Operators in Harmonic Analysis*, Cambridge University Press, Cambridge, 1991.
- [20] A.W. Knap, E.M. Stein, *Intertwining operators for semisimple groups*, Annals of Mathematics, Vol. 93, No.3, 1971, pp. 489-578
- [21] P. Van Lancker, F. Sommen, D. Constaes, *Models for irreducible representations of $Spin(m)$* , Advances in Applied Clifford Algebras, Vol. 11, Issue 1 supplement, 2001, pp. 271-289.

- [22] B. Lawson, M.L. Michelson, *Spin Geometry*, Princeton University Press, Princeton, New Jersey, 1989.
- [23] J. Li, J. Ryan, *Some operators associated to Rarita-Schwinger type operators*, Complex Variables and Elliptic Equations: An International Journal, Volume 57, Issue 7-8, 2012, pp. 885-902.
- [24] M. Mitrea, *Singular Integrals, Hardy Spaces and Clifford Wavelets*, Lecture Notes in Mathematics, No. 1575, Springer Verlag, Heidelberg, 1994.
- [25] J. Peetre, T. Qian, *Möbius covariance of iterated Dirac operators*, J. Austral. Math. Soc. Series A 56 (1994), pp. 403-414.
- [26] I. Porteous, *Clifford algebra and the classical groups*, Cambridge University Press, Cambridge, 1995.
- [27] W. Rarita, J. Schwinger, *On a Theory of Particles with Half-integral Spin*, Phys. Rev., Vol. 60, Issue 1, 1941, pp. 60-61.
- [28] J. Ryan, *Dirac Operators in Analysis and Geometry*, Lecture note, 2008. <http://comp.uark.edu/~jryan/notes.doc>
- [29] J. Ryan, *Conformally covariant operators in Clifford analysis*, Z. Anal. Anwendungen, 14, 1995, pp. 677-704.
- [30] J. Ryan, *Iterated Dirac operators and conformal transformations in \mathbb{R}^m* , Proceedings of the XV International Conference on Differential Geometric Methods in Theoretical Physics, World Scientific, 1987, pp. 390-399.
- [31] H. De Schepper, D. Eelbode, T. Raeymaekers, *On a special type of solutions of arbitrary higher spin Dirac operators*, J. Phys. A: Math. Theor., 43, 2010, 325208-325221.
- [32] E. Stein, G. Weiss, *Generalization of the Cauchy-Riemann equations and representations of the rotation group*, Amer. J. Math. 90 (1968), pp. 163-196.
- [33] S. Shirrell, R. Walter, *Hermitian Clifford Analysis and Its Connections with Representation Theory*, to appear in Complex Variables and Elliptic Equations.
- [34] J. Slovák, *Natural Operators on Conformal Manifolds*, Habilitation thesis, Masaryk University, Brno, Czech Republic, 1993.
- [35] V. Souček, *Higher spins and conformal invariance in Clifford analysis*, Proc. Conf. Seiffen. 1996. pp. 175-185.